

Sample Solution for Sample Question 13

1. To show that $(\frac{1}{\sqrt{n}} Q^*) (\frac{1}{\sqrt{n}} Q) = I$,

we show $Q^* Q = nI$ instead.

The j,k -entry of $Q^* Q$ is

$$\sum_{l=1}^n (Q^*)_{j,l} (Q)_{l,k} = \sum_{l=1}^n \overline{\omega^{(l-1)(j-1)}} \cdot \omega^{(l-1)(k-1)}$$

$$= \sum_{l=1}^n \omega^{-(l-1)(j-1) + (l-1)(k-1)} = \sum_{l=1}^n \omega^{(k-j)(l-1)}$$

$$= \sum_{l=1}^n r^{l-1}, \quad \text{where } r = \omega^{k-j}.$$

$$= \begin{cases} n & \text{if } r=1 \\ \frac{r^n - 1}{r-1} & \text{if } r \neq 1 \end{cases}$$

$$= \begin{cases} n & \text{if } k=j \\ 0 & \text{if } k \neq j \end{cases} \quad (\text{since } r^n = \omega^{(k-j) \cdot n} = (\omega^n)^{k-j} = 1.)$$

$$2. A = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}.$$

$$S_0 = 1$$

$$S_1 = 3 + 4 = 7$$

$$S_2 = 3 \cdot 4 - 5 \cdot 2 = 2$$

$$\begin{aligned} \Rightarrow p(x) &= S_0(-x)^2 + S_1(-x)^1 + S_2(-x)^0 \\ &= x^2 - 7x + 2. \end{aligned}$$

$$3. A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$S_0 = 1$$

$$S_1 = 1 + 2 + 9 = 12$$

$$S_2 = 1 + 8 + 6 = 15$$

$$S_3 = \det(A) = 2$$

$$\alpha \quad \{1, 2\} \quad \{1, 3\} \quad \{2, 3\}$$

$$A[\alpha] \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix} \quad \begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix}$$

$$\det \quad 1 \quad 8 \quad 6$$

$$\begin{aligned} \Rightarrow p(x) &= S_0(-x)^3 + S_1(-x)^2 + S_2(-x)^1 + S_3(-x)^0 \\ &= -x^3 + 12x^2 - 15x + 2. \end{aligned}$$

4. Observe that

$$\det(J_k) = \begin{cases} 0 & \text{if } k \geq 2 \\ 1 & \text{if } k = 1. \end{cases}$$

$$S_0 = 1$$

$$S_1 = \underbrace{1+1+\dots+1}_{n \text{ 1's}} = n$$

$$S_2 = \dots = S_n = 0.$$

$$\begin{aligned} \Rightarrow p(x) &= S_0(-x)^n + S_1(-x)^{n-1} + \dots + S_n(-x)^0 \\ &= (-1)^n x^n + (-1)^{n-1} \cdot n \cdot x^{n-1} \\ &= (-1)^n x^{n-1} \cdot (x - n). \end{aligned}$$

5. Recall that $\det(J_n - xI_n) = (-1)^n x^{n-1} \cdot (x - n)$.

Thus,

$$\begin{aligned} \det(J_n - I_n - xI_n) &= \det(J_n - (x+1)I_n) \\ &= (-1)^n (x+1)^{n-1} (x+1-n). \end{aligned}$$

用 $x+1$
代 x .

6. Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

Then

$$p(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_0 I$$

$$\begin{aligned} P(Q^{-1}AQ) &= a_n (Q^{-1}AQ)^n + a_{n-1} (Q^{-1}AQ)^{n-1} + \dots + a_0 I \\ &= a_n Q^{-1} A^n Q + a_{n-1} Q^{-1} A^{n-1} Q + \dots + a_0 I \\ &= Q^{-1} (a_n A^n + a_{n-1} A^{n-1} + \dots + a_0 Q) \\ &= Q^{-1} (a_n A^n + a_{n-1} A^{n-1} + \dots + a_0) Q \\ &= Q^{-1} p(A) Q. \end{aligned}$$

7. Let A be a diagonalizable

and ~~supp~~ assume $Q^{-1}AQ = D$,

where D is a diagonal matrix.

Since $A \sim D$, they have the same

similar

characteristic polynomial $p(x)$.

We may assume $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$\text{and } p(x) = (-1)^n (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

Check that

$$p(D) = (-1)^n (D - \lambda_1 I)(D - \lambda_2 I) \cdots (D - \lambda_n I).$$

$$= (-1)^n \begin{pmatrix} 0 & & & \\ \lambda_2 - \lambda_1 & & & \\ & \lambda_3 - \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n - \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1 - \lambda_2 & & & \\ & 0 & & \\ & & \lambda_3 - \lambda_2 & \\ & & & \ddots \\ & & & & \lambda_n - \lambda_2 \end{pmatrix} \cdots \begin{pmatrix} \lambda_1 - \lambda_n & & & \\ & \lambda_2 - \lambda_n & & \\ & & \lambda_3 - \lambda_n & \\ & & & \ddots \\ & & & & 0 \end{pmatrix}$$

$$= 0 \quad (\text{對角線矩陣相乘})$$

等於對角線上各相對應項相乘)

By Problem 7,

$$Q^{-1}p(A)Q = p(Q^{-1}AQ) = p(D) = 0.$$

$$\Rightarrow p(A) = Q0Q^{-1} = 0.$$

