

Sample Solutions for Sample Questions 2.

$$1. \quad M = \text{Rep}_{\mathcal{B}, \mathcal{D}}(\text{id}) = \begin{pmatrix} | & | & | \\ \text{Rep}_{\mathcal{D}}(1) & \text{Rep}_{\mathcal{D}}(x) & \text{Rep}_{\mathcal{D}}(x^2) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The entries come from:

$$1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x(x-1)$$

$$x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x(x-1)$$

$$x^2 = 0 \cdot 1 + 1 \cdot x + 1 \cdot x(x-1)$$

$$N = \text{Rep}_{\mathcal{D}, \mathcal{B}}(\text{id}) = \begin{pmatrix} | & | & | \\ \text{Rep}_{\mathcal{B}}(1) & \text{Rep}_{\mathcal{B}}(x) & \text{Rep}_{\mathcal{B}}(x(x-1)) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

The entries come from:

$$1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$$

$$x(x-1) = 0 \cdot 1 - 1 \cdot x + 1 \cdot x^2$$

$$\text{Check: } MN = NM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{True.}$$

2.

$$M = \text{Rep}_{B,D}(\text{id}) = \left(\begin{array}{c|c|c} \text{Rep}_D \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \text{Rep}_D \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \text{Rep}_D \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \right) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The entries come from:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$N = \text{Rep}_{\mathcal{D}, B}(\text{id}) = \left(\begin{array}{c|c|c} \text{Rep}_B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \text{Rep}_B \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} & \text{Rep}_B \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{array} \right) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The entries come from:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Check: } MN = NM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

True.

⊗ Before Problem 3, let's find the relations between $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix}$

$$\textcircled{1} \text{ Let } M = \text{Rep}_{\mathcal{D}, \mathcal{D}}(\text{id}) = \begin{pmatrix} \text{Rep}_{\mathcal{D}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{Rep}_{\mathcal{D}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

The entries come from:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \left(-\frac{1}{\sqrt{2}}\right) \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\text{Then } M \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow \begin{cases} x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ y' = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{cases} \dots (1)$$

$$\textcircled{2} \text{ Let } N = \text{Rep}_{\mathcal{D}, \mathcal{B}}(\text{id}) = \begin{pmatrix} \text{Rep}_{\mathcal{B}}(\text{id}) & \end{pmatrix} = \begin{pmatrix} \text{Rep}_{\mathcal{B}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} & \text{Rep}_{\mathcal{B}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\text{Rep}_{\mathcal{B}}(\vec{v}) = \vec{v} \text{ for all } \vec{v} \in \mathcal{R}^2$$

since \mathcal{B} is the standard basis

$$\text{Then } N \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y' \\ y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \end{cases} \dots (2)$$

3. Use Equation (2) on page 3.

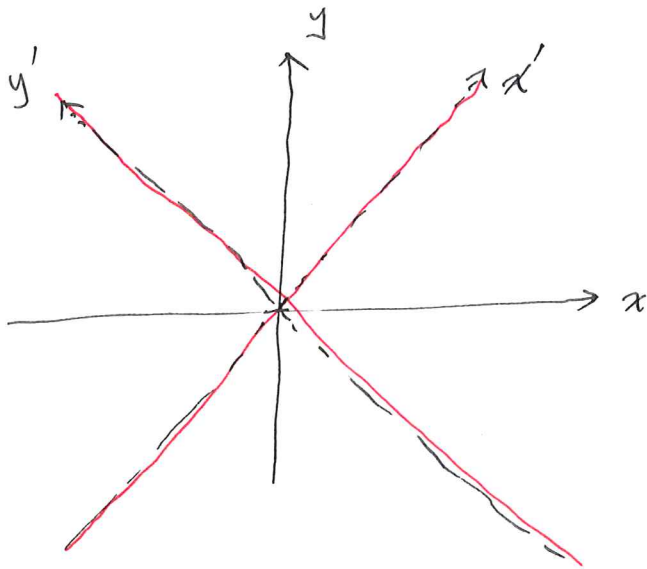
$$x+y = \left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right) + \left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right) = \sqrt{2}x'$$

$$x-y = \left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right) - \left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right) = -\sqrt{2}y'$$

So the new equation is $(\sqrt{2}x')(-\sqrt{2}y') = 0$

$$\Downarrow$$

$$-2x'y' = 0.$$



The red figure is
 $(x+y)(x-y) = 0$
 on xy -plane,
 and is
 $-2x'y' = 0$
 on $x'y'$ -plane.

4. Again, use Equation (2) on page 3.

$$x^2 + y^2 = \left(\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2\right) + \left(\frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2\right)$$

$$= x'^2 + y'^2$$

(not surprising, since two coordinating systems has the same radius.)

$$xy = \frac{1}{2}x'^2 - \frac{1}{2}y'^2$$

So $7x^2 - 2xy + 7y^2 = 1$ becomes $7(x'^2 + y'^2) - 2\left(\frac{1}{2}x'^2 - \frac{1}{2}y'^2\right) = 1$.

$$\Leftrightarrow 6x'^2 + 8y'^2 = 1.$$

Use desmos.com to see the ellipse.

5.

By definition,

$$\text{Rep}_{\mathcal{D}, \mathcal{D}}(f) = \begin{pmatrix} \text{Rep}_{\mathcal{D}} \left(f \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right) & \text{Rep}_{\mathcal{D}} \left(f \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right) \\ 1 & 1 \end{pmatrix}$$

Computation:

$$f \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = J_2 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} = 2 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + 0 \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

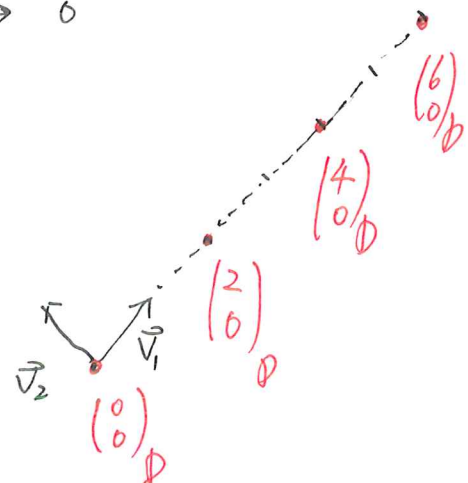
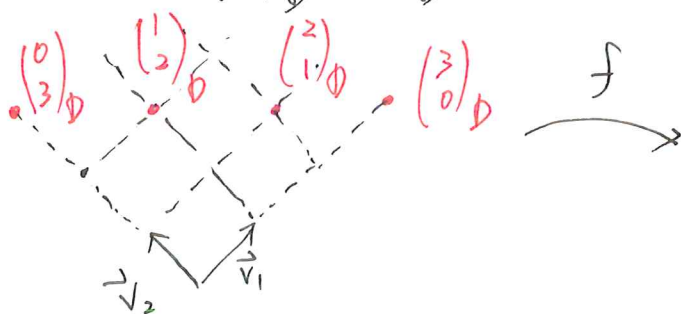
$$f \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = J_2 \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + 0 \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow \text{Rep}_{\mathcal{D}, \mathcal{D}}(f) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \Lambda$$

Let $\vec{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\text{Rep}_{\mathcal{D}, \mathcal{D}}(f) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{D}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}_{\mathcal{D}}$ means $\vec{v}_1 \xrightarrow{f} 2\vec{v}_1$

$\text{Rep}_{\mathcal{D}, \mathcal{D}}(f) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{D}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{\mathcal{D}}$ means $\vec{v}_2 \xrightarrow{f} \vec{0}$



6.

In fact, we already computed

$$Q = \text{Rep}_{\mathcal{D}, S_2}(\text{id}) = N \text{ on page 3.}$$

$$Q^{-1} = \text{Rep}_{S_2, \mathcal{D}}(\text{id}) = M \text{ on page 3.}$$

You may check

$$Q^{-1} J_2 Q = \Lambda \text{ on page 5.}$$

This is not a coincidence.

$$\Lambda = \text{Rep}_{\mathcal{D}, \mathcal{D}}(f)$$

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$$\underbrace{\text{Rep}_{S_2, \mathcal{D}}(\text{id})}_{Q^{-1}} \cdot \underbrace{\text{Rep}_{S_2, S_2}(f)}_{J_2} \cdot \underbrace{\text{Rep}_{\mathcal{D}, S_2}(\text{id})}_Q$$

7.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \\ 4 & 6 & 10 \end{pmatrix} \xrightarrow[\substack{-3r_1+r_2 \\ -4r_1+r_3}]{-3r_1+r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{-2c_1+c_2 \\ -3c_1+c_3}]{-c_2+c_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-c_2+c_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = B.$$

$\xleftarrow{-\frac{1}{2}r_2}$ $\xleftarrow{-r_2+r_3}$ $\xleftarrow[\substack{-3r_1+r_2 \\ -4r_1+r_3}]{-c_2+c_3}$

Let $P = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1 & & \\ -3 & 1 & \\ -4 & & 1 \end{pmatrix}$

注意順序.

$Q = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

You may compute $P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ -1 & -1 & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

and check $PAQ = B.$

↔ 愈接近 A 的矩陣愈先發生.

