Sample Questions 5

1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{u} = \mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in \operatorname{span}\{\mathbf{v}\}$ and $\langle \mathbf{v}, \mathbf{y} \rangle = 0$.

2. Suppose $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times \ell$ matrix. Show that $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$. That is, show that the i, j-entry of $(\mathbf{AB})^\top$ and the i, j-entry of $\mathbf{B}^\top \mathbf{A}^\top$ are the same for $i = 1, \ldots, m$ and $j = 1, \ldots, \ell$.

3. Let
$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$
Compute $\langle \mathbf{Ax}, \mathbf{y} \rangle$ and $\langle \mathbf{x}, \mathbf{A}^\top \mathbf{y} \rangle$ separately and check if they are the same.

4. Show that $\mathbf{Ax} = \mathbf{0}$ if and only if $\mathbf{A}^\top \mathbf{Ax} = \mathbf{0}$. [One direction is easy while the other is tricky. Hint: Suppose $\mathbf{A}^\top \mathbf{Ax} = \mathbf{0}$. then $\langle \mathbf{x}, \mathbf{A}^\top \mathbf{Ax} \rangle = 0$ and you can move $\mathbf{A}^\top$ to the other side.]

5. Show that $\mathbf{A}$ has full column rank if and only if $\mathbf{A}^\top \mathbf{A}$ is invertible.

6. Let
$$\mathbf{x} = \begin{bmatrix} 3+2i \\ 2-3i \\ i \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 3+4i \\ -4i \\ 2-i \end{bmatrix} \in \mathbb{C}^3.$$
Find $\langle \mathbf{x}, \mathbf{y} \rangle$, $\langle \mathbf{y}, \mathbf{x} \rangle$, and $|\mathbf{x}|$ (where the inner product is over $\mathbb{C}$).

7. A Vandermonde matrix is of the form
$$\mathbf{M}(p_1, \ldots, p_n) = \begin{bmatrix} 1 & p_1 & \cdots & p_1^{n-1} \\ 1 & p_2 & \cdots & p_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_n & \cdots & p_n^{n-1} \end{bmatrix}.$$
Suppose a polynomial $f(x) = a + bx + cx^2 + dx^3 + ex^4$ passes through the five points $(p_1, q_1), \ldots, (p_5, q_5)$. Show that
$$\mathbf{M}(p_1, p_2, p_3, p_4, p_5) \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix}.$$
[Therefore, you can use five points to determine a degree-four polynomial.]