Sample ~~Questions~~ Solutions for Sample Questions 7

1. Gram-Schmidt Algorithm

$$\begin{cases} \vec{u}_1 = \vec{v}_1 \\ \vec{u}_2 = \vec{v}_2 - \mathrm{proj}_{[\vec{u}_1]}(\vec{v}_2) \\ \vec{u}_3 = \vec{v}_3 - \mathrm{proj}_{[\vec{u}_1]}(\vec{v}_3) - \mathrm{proj}_{[\vec{u}_2]}(\vec{v}_3). \end{cases}$$

Recall that $\mathrm{proj}_{[u]}(\vec{v}) = \dfrac{\langle \vec{v}, \vec{u} \rangle}{|\vec{u}|^2} \cdot \vec{u}$.

So $\vec{u}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

[You may now check $\langle \vec{u}_1, \vec{u}_2 \rangle = 0$.]

$$\vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1/3}{6/9} \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/6 \\ 2/6 \\ 2/6 \end{pmatrix} - \begin{pmatrix} -2/6 \\ 1/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$$

[You may check $\langle \vec{u}_1, \vec{u}_3 \rangle = \langle \vec{u}_2, \vec{u}_3 \rangle = 0$.]

2. Normalize : $\hat{u} = \frac{1}{|\vec{u}|}\cdot\vec{u}$

So $\hat{u}_1 = \frac{1}{|\vec{u}_1|}\cdot\vec{u}_1 = \frac{1}{\sqrt{3}}\vec{u}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \rightarrow \vec{u}_1 = \sqrt{3}\,\hat{u}_1$

$\hat{u}_2 = \frac{1}{|\vec{u}_2|}\cdot\vec{u}_2 = \frac{1}{\sqrt{2/3}}\vec{u}_2 = \begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \rightarrow \vec{u}_2 = \sqrt{\frac{2}{3}}\,\hat{u}_2$

$\hat{u}_3 = \frac{1}{|\vec{u}_3|}\cdot\vec{u}_3 = \frac{1}{\sqrt{1/2}}\vec{u}_3 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \rightarrow \vec{u}_3 = \frac{1}{\sqrt{2}}\,\hat{u}_3$

Exam the caloulation :

$\vec{u}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \vec{u}_1 = \cancel{\sqrt{3}\,\vec{u}_1}\ \underline{\sqrt{3}\,\hat{u}_1 + 0\,\hat{u}_2 + 0\,\hat{u}_3}$

$\vec{u}_2 = \vec{v}_2 - \frac{2}{3}\vec{u}_1 \Rightarrow \vec{v}_2 = \frac{2}{3}\vec{u}_1 + \vec{u}_2 + 0\,\vec{u}_3$

$= \underline{\frac{2}{\sqrt{3}}\hat{u}_1 + \sqrt{\frac{2}{3}}\,\hat{u}_2 + 0\,\hat{u}_3}$

$\vec{u}_3 = \vec{v}_3 - \frac{1}{3}\vec{u}_1 - \frac{1}{2}\vec{u}_2 \Rightarrow \vec{v}_3 = \frac{1}{3}\vec{u}_1 + \frac{1}{2}\vec{u}_2 + \vec{u}_3$

$= \underline{\frac{1}{\sqrt{3}}\hat{u}_1 + \frac{1}{\sqrt{6}}\hat{u}_2 + \frac{1}{\sqrt{2}}\hat{u}_3}$.

So $A = Q \cdot R$

$$\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ | & | & | \end{pmatrix}\begin{pmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

orthogonal

since $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ is orthonormal

upper triangular.

3. Find a basis for V first.

V is the null space of $[1 \ 1 \ 1]$

So $V = \text{span}\left\{ \underbrace{\overset{\vec{v}_1}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}, \overset{\vec{v}_2}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}}_{\text{找零解.}} \right\}$

Apply Gram-Schmidt,

$$\vec{u}_1 = \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{[\vec{u}_1]}(\vec{v}_2) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Then $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis.

[If you want orthonormal basis, then normalize $\vec{u}_1$ and $\vec{u}_2$].

[Of course you can cheat in this special case.

That is, ~~the~~ take $\vec{u}_2$ and $\vec{u}_3$ as in Problem 1.]

4.

Method 1.

Let $A = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{pmatrix}$. Then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal basis.

In this case, $Rep_B(\vec{v}) = \begin{pmatrix} \langle \vec{v}, \vec{u}_1 \rangle \\ \vdots \\ \langle \vec{v}, \vec{u}_n \rangle \end{pmatrix}$

$$= \begin{pmatrix} — & \vec{u}_1 & — \\ & \vdots & \\ — & \vec{u}_n & — \end{pmatrix} \begin{pmatrix} | \\ \vec{v} \\ | \end{pmatrix} = A^T \vec{v}.$$

Method 2.

$Rep_B(\vec{v}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \Leftrightarrow \vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$

$\Leftrightarrow \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{v}$.

So, $A \cdot Rep_B(\vec{v}) = \vec{v}$

Since $AA^T = A^TA = I \Rightarrow A^T = A^{-1}$.

$\Rightarrow Rep_B(\vec{v}) = A^{-1}\vec{v} = A^T\vec{v}$.

5.

$$A \xrightarrow[\times 1]{\substack{-r_1+r_2 \\ -r_1+r_3 \\ -r_1+r_4}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix} \xrightarrow[\times 1]{-r_3+r_4} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & 2 & \\ & 1 & 1 & 1 \\ & & 2 & 1 \end{pmatrix}$$

$$\det = -2 \longleftarrow \det = -2 \longleftarrow \det = -2 \longleftarrow$$

$$\xrightarrow[\times 1]{-r_2+r_4} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & 2 & \\ & 1 & 1 & 1 \\ & & & 1 \end{pmatrix} \xrightarrow[\times(-1)]{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & 1 & 1 & 1 \\ & & 2 & \\ & & & 1 \end{pmatrix}$$

$$\longleftarrow \det = -2 \longleftarrow \det = 2 \longleftarrow$$

$$\xrightarrow[\times \frac{1}{2}]{\frac{1}{2} r_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & 1 & 1 & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix} \xrightarrow[\times 1]{\text{lots of row operation}} \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$\longleftarrow \det = 1 \longleftarrow \det = 1$$

So $\det(A) = -2$.

6. Type 1. $r_i \leftrightarrow r_j$

$$E = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{matrix}, \quad E \xrightarrow{r_i \leftrightarrow r_j} I_n$$

$$\begin{matrix} \uparrow & \uparrow \\ i & j \end{matrix}$$

So $\det(E) = -\det(I_n) = -1$.

$E^{-1} = E \Rightarrow \det(E^{-1}) = -1 = \det(E)^{-1}$

Type 2. kr_i

$$E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & k & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow i \\ \\ \\ \end{matrix}, \quad E \xrightarrow{\frac{1}{k} r_i} I_n$$

$$\begin{matrix} \uparrow \\ i \end{matrix}$$

So $\frac{1}{k}\det(E) = \det(I_n) = 1 \Rightarrow \det(E) = k$.

E^{-1} is the elementary matrix from $\frac{1}{k} r_i$, so $\det(E^{-1}) = \frac{1}{k} = \det(E)^{-1}$

Type 3. $kr_i + r_j$

$$E = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & k & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{matrix} \\ \leftarrow i \\ \leftarrow j \\ \\ \\ \end{matrix}, \quad E \xrightarrow{-kr_i + r_j} I_n.$$

So $\det(E) = \det(I_n) = 1$.

E^{-1} is the elementary matrix from $-kr_i + r_j$,

so $\det(E^{-1}) = 1 = \det(E)$.

7. By recording the elimination process, we can find

$$E_k \cdots E_2 E_1 A = R$$ or

$$A = E_1^{-1} \cdots E_k^{-1} R.$$

By observations in Problem 6,

$$\det(E_k) \cdots \det(E_1) \cdot \det(A) = \det(R).$$

$$\Updownarrow$$

$$\det(A) = \det(E_1)^{-1} \cdots \det(E_k)^{-1} \det(R)$$
$$= \det(E_1^{-1}) \cdots \det(E_k^{-1}) \cdot \det(R).$$

Note that $\det(E_1), \ldots, \det(E_k)$ are not zero, since the determinant of an elementary matrix is never zero by Problem 6.

$\det(A) \neq 0$ if and only if $\det(R) \neq 0$

if and only if ~~A is inver~~ R has no zero rows

if and only if $R = I_n$

if and only if A is invertible.