

Sample Solutions for Sample Questions 9

1. Let $A = \begin{pmatrix} \text{---} \vec{r}_1 \text{---} \\ \vdots \\ \text{---} \vec{r}_n \text{---} \end{pmatrix}$.

Then $-A = \begin{pmatrix} \text{---} -\vec{r}_1 \text{---} \\ \vdots \\ \text{---} -\vec{r}_n \text{---} \end{pmatrix}$.

Thus, $\det(-A) = (-1) \cdot \det \begin{pmatrix} \text{---} -r_1 \text{---} \\ \text{---} -r_2 \text{---} \\ \vdots \\ \text{---} -r_n \text{---} \end{pmatrix}$
 $= (-1)^2 \cdot \det \begin{pmatrix} \text{---} r_1 \text{---} \\ \text{---} r_2 \text{---} \\ \vdots \\ \text{---} r_n \text{---} \end{pmatrix}$
 $= (-1)^n \cdot \det(A)$.

Suppose A is an ^{an} **skew** ~~sym~~-symmetric matrix.

Then $A^T = -A$.

When n is odd,

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$$

$$\Rightarrow \det(A) = 0$$

$\Rightarrow A$ is singular.

2.

Suppose A is an $n \times n$ orthogonal matrix.

$$\text{Then } AA^T = I_n \Rightarrow \det(A) \cdot \det(A^T) = \det(I_n) = 1.$$

$$\text{Since } \cancel{A} \Rightarrow \det(A) \cdot \det(A) = 1$$

$$\Rightarrow \det(A) = \pm 1$$

$$\text{and } |\det(A)| = 1.$$

Suppose $B = \begin{pmatrix} \text{---} \vec{v}_1 \text{---} \\ \vdots \\ \text{---} \vec{v}_n \text{---} \end{pmatrix}$ such that

the rows are mutually orthogonal.

⊙ If some of the rows are zero,

then

$$\det(B) = 0 = |\vec{v}_1| \cdots |\vec{v}_n|$$

⊙ If each row is nonzero,

$$\text{write } \vec{u}_i = \frac{\vec{v}_i}{|\vec{v}_i|} \text{ and } \vec{v}_i = |\vec{v}_i| \cdot \vec{u}_i.$$

$$\begin{aligned} \text{Thus, } \det(B) &= \det \begin{pmatrix} \text{---} |\vec{v}_1| \cdot \vec{u}_1 \text{---} \\ \vdots \\ \text{---} |\vec{v}_n| \cdot \vec{u}_n \text{---} \end{pmatrix} \\ &= |\vec{v}_1| \cdots |\vec{v}_n| \cdot \det \begin{pmatrix} \text{---} \vec{u}_1 \text{---} \\ \vdots \\ \text{---} \vec{u}_n \text{---} \end{pmatrix} \end{aligned}$$

$$\Rightarrow |\det(B)| = |\vec{v}_1| \cdots |\vec{v}_n|$$

$\det = \pm 1$ since orthogonal.

3.

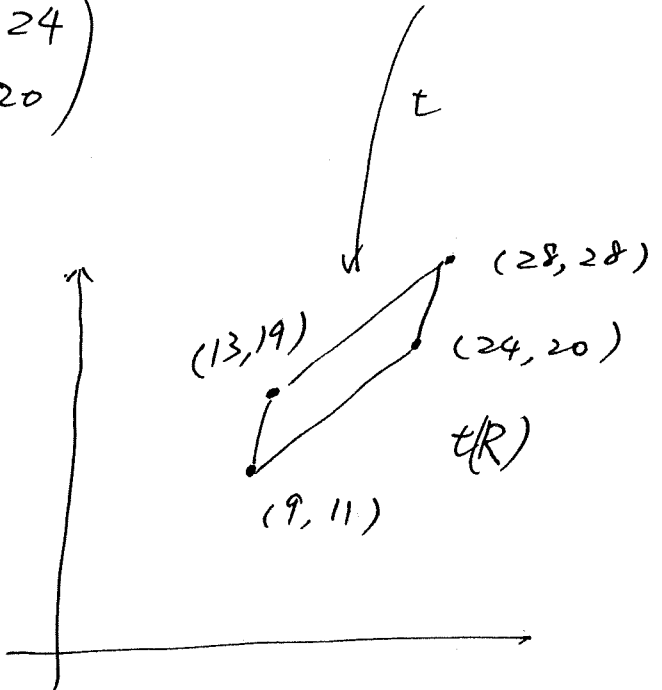
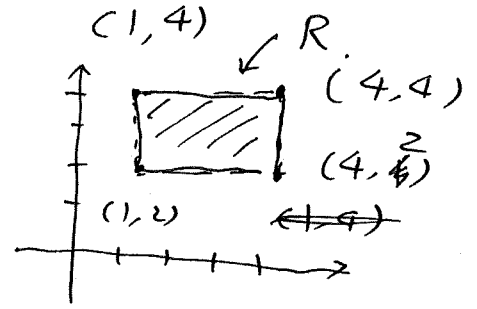
Compute

$$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 11 \end{pmatrix}$$

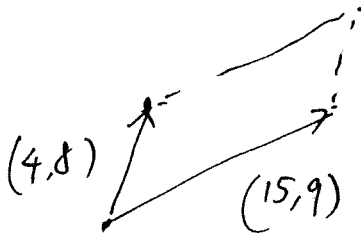
$$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 22 \\ 16 \end{pmatrix} \begin{pmatrix} 24 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 13 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 28 \\ 28 \end{pmatrix}$$



← 平移



compute $\begin{vmatrix} 15 & 9 \\ 4 & 8 \end{vmatrix} = 120 - 36 = 84.$

area = 84.

Alternatively, the area of $R = 3 \cdot 2 = 6.$

$$\text{area of } t(R) = \left| \det(A) \cdot 6 \right| = 14 \cdot 6 = 84.$$

4.

$$\det \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix} + [-(-1)] \det \begin{pmatrix} -1 & -1 & & \\ & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

$$= 4 \cdot \det \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{pmatrix} - \det \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix}$$

$$= 3 \cdot \det \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{pmatrix}$$

now you may use 3x3 determinant formula

$$= 3 \cdot (2 \cdot 2 \cdot 2 - 2 - 2) + 2 \cdot (-4 + 1)$$

$$= 12 - 6$$

$$= 6.$$

5. Let $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{pmatrix}$ \leftarrow $n \times n$ matrix.

If $A \cdot \vec{1} = \vec{0} \Rightarrow \vec{v}_1 + \dots + \vec{v}_n = \vec{0}$.

Let \vec{u}_i be a vector in \mathbb{R}^{n-1} obtained from \vec{v}_i by removing the first entry.

Then $A(1,1) = \begin{pmatrix} | & & | \\ \vec{u}_2 & \vec{u}_3 & \dots & \vec{u}_n \\ | & & & | \end{pmatrix}$ \leftarrow $(n-1) \times (n-1)$

$A(1,2) = \begin{pmatrix} | & & | \\ \vec{u}_1 & \vec{u}_3 & \dots & \vec{u}_n \\ | & & & | \end{pmatrix}$.

Also, $\vec{u}_1 + \dots + \vec{u}_n = \vec{0} \iff \vec{u}_1 = -(\vec{u}_2 + \dots + \vec{u}_n)$.

Thus,

$\det(A(1,1)) = \det \begin{pmatrix} | & & | \\ \vec{u}_2 & \vec{u}_3 & \dots & \vec{u}_n \\ | & & & | \end{pmatrix}$

$= \det \begin{pmatrix} | & & | \\ \vec{u}_2 + \dots + \vec{u}_n & \vec{u}_3 & \dots & \vec{u}_n \\ | & & & | \end{pmatrix}$

$= \det \begin{pmatrix} | & & | \\ -\vec{u}_1 & \vec{u}_3 & \dots & \vec{u}_n \\ | & & & | \end{pmatrix} = -\det(A(1,2))$

$$6. \quad X = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \vdots & & \\ & & & 1 & \\ & & & & 1 \\ \hline & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \vdots & \\ & & & & 1 \\ & & & & 2 \\ \hline & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \vdots & \\ & & & & 2 \\ & & & & 3 \end{pmatrix}$$

$$BA^{-1}B \cancel{X/A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$X/A = C - BA^{-1}B$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \det(X) = \det(A) \cdot \det(X/A)$$

$$= 1 \cdot (-1) = -1$$

$$J_n - I_n = \begin{pmatrix} 0 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ 1 & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

~~det~~ \overline{J}

$$\det(J_n - I_n) = \det \begin{pmatrix} n-1 & & & & 1 \\ n-1 & 0 & & & \\ \vdots & & \ddots & & \\ n-1 & 1 & & \ddots & 0 \end{pmatrix}$$

(所有 column 加到第一个 column)

$$= (n-1) \cdot \det \begin{pmatrix} 1 & & & & 1 \\ 1 & 0 & & & \\ \vdots & & \ddots & & \\ 1 & 1 & & \ddots & 0 \end{pmatrix}$$

(提出 column 1 的 $n-1$)

$$= \det (n-1) \cdot \det \begin{pmatrix} 1 & & & & 0 \\ \vdots & -1 & & & \\ \vdots & & \ddots & & \\ \vdots & 0 & & \ddots & \\ \vdots & & & & -1 \end{pmatrix}$$

(把 $-1 \times$ (column 1) 加到每个 column)

$$= (-1)^{(n-1)} \cdot (n-1)$$

(下三角矩阵)

