國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

期末考 January 10, 2022 **Final Exam**

姓名 Name : Solution

學號 Student ID # : ______

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Find subsets $A, B \subseteq \{0, 2, 4, 6, 8\}$ and a function $f : A \to B$ such that f is injective but not surjective.

Let $A = \{0, 2\}$ $\quad f: A \to B$
$B = \{0, 2, 4\}$. $\quad f(x) = x$.

2. [1pt] Find subsets $A, B \subseteq \{0, 2, 4, 6, 8\}$ and a function $f : A \to B$ such that f is surjective but not injective.

Let $A = \{0, 2, 4\}$ $\quad f: A \to B$
$B = \{0, 2\}$
$0 \mapsto 0$
$2 \mapsto 2$
$4 \mapsto 2$.

3. [1pt] Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ be the set of all nonnegative integers and $\mathbb{Z}$ the set of all integers. Find a bijection from $\mathbb{N}_0$ to $\mathbb{Z}$.

Let $f: \mathbb{N}_0 \to \mathbb{Z}$ such that $f(0) = 0$
$f(n) = \frac{n+1}{2}$ if n odd
$f(n) = -\frac{n}{2}$ if n even.

4. [1pt] Find a surjective linear function $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(1, 1, 1) = (0, 0)$.

Let $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and let $f = f_A$.

5. [1pt] Find an injective linear function $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(1, 0) = (1, 1, 1)$.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and let $f = f_A$.

6. Let

$$\mathbf{u}_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix}0\\0\\-1\end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix},$$

and $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis of $\mathbb{R}^3$. Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear function such that

$$f(\mathbf{u}_1) = \begin{bmatrix}1\\4\end{bmatrix}, \quad f(\mathbf{u}_2) = \begin{bmatrix}2\\5\end{bmatrix}, \quad \text{and } f(\mathbf{u}_3) = \begin{bmatrix}3\\6\end{bmatrix}.$$

(a) [1pt] Find $f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)$.

$$f\begin{pmatrix}0\\1\\1\end{pmatrix} = f(u_1 - u_2) = f(u_1) - f(u_2) = \begin{pmatrix}1\\4\end{pmatrix} - \begin{pmatrix}2\\5\end{pmatrix} = \underline{\begin{bmatrix}-1\\-1\end{bmatrix}}$$

(b) [1pt] Find $f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$.

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = f(-u_1 + u_2 + u_3) = f(u_1) + f(u_2) + f(u_3) = -\begin{pmatrix}1\\4\end{pmatrix} + \begin{pmatrix}2\\5\end{pmatrix} + \begin{pmatrix}3\\6\end{pmatrix}$$

$$= \underline{\begin{pmatrix}4\\7\end{pmatrix}}$$

(c) [~~2~~ 3pt] Find a matrix A such that $f(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^3$.

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{bmatrix}4\\7\end{bmatrix}$$

$$f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{bmatrix}1\\4\end{bmatrix} \qquad \Rightarrow \underline{A = \begin{bmatrix}4 & 1 & -2\\7 & 4 & -5\end{bmatrix}}$$

$$f\begin{pmatrix}0\\0\\1\end{pmatrix} = -f(u_2) = -\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}-2\\-5\end{bmatrix}$$

7. Let

$$\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix}1\\-1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}.$$

Let $\alpha = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis of $\mathbb{R}^3$ and $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ a basis of $\mathbb{R}^2$. Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear function such that

$$f(\mathbf{u}_1) = \mathbf{v}_1,$$
$$f(\mathbf{u}_2) = \mathbf{v}_2,$$
$$f(\mathbf{u}_3) = \mathbf{v}_1.$$

Recall that $\mathcal{E}_n$ is the standard basis of $\mathbb{R}^n$.

(a) [1pt] Find the matrix representation $[f]_\alpha^\beta$.

$$[f]_\alpha^\beta = \begin{bmatrix} | & | & | \\ [f(u_1)]_\beta & [f(u_2)]_\beta & [f(u_3)]_\beta \\ | & | & | \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

(b) [2pt] Find the change of basis matrices $[\mathrm{id}]_\alpha^{\mathcal{E}_3}$ and $[\mathrm{id}]_{\mathcal{E}_3}^\alpha$.

$$[id]_\alpha^{\mathcal{E}_3} = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} = \underline{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \quad [id]_{\mathcal{E}_3}^\alpha = \left([id]_\alpha^{\mathcal{E}_3}\right)^{-1} = \underline{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

(c) [2pt] Find a matrix A such that $f(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^3$.

$$A = [f] = [id]_\beta^{\mathcal{E}_2} [f]_\alpha^\beta [id]_{\mathcal{E}_3}^\alpha$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \end{bmatrix}}$$

8. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是線性函數（linear function）。

   請寫下線性函數的定義，並說明它和矩陣之間的關係。請以自己的方式、盡量白話的敘述、或是比喻來說明這個爲什麼要這樣定義？還有爲什麼要考慮這樣的概念？請給一些能幫助他人理解的例子（正面的、反面的），並提出一些這個概念的相關性質；有必要的話可以加上一些圖來輔助說明。格式沒有限制，篇幅大約半面到一面。

   (If Chinese is not your native language, you may use English or the language that you prefer.)

9. [extra 5pt] Let $\beta = \{\mathbf{u}_0, \ldots, \mathbf{u}_k\}$ be a set of nonzero vectors in $\mathbb{R}^n$. Suppose $\lambda_0, \ldots, \lambda_k$ are distinct real numbers and A is an $n \times n$ matrix such that $A\mathbf{u}_i = \lambda_i \mathbf{u}_i$ for $i = 0, \ldots, k$. Show that β is linearly independent.

Suppose $c_0\vec{u}_0 + c_1\vec{u}_1 + \cdots + c_k\vec{u}_k = \vec{0}$ ... (0)

Then by multiplying A on both sides,

we have $c_0\lambda_0\vec{u}_0 + c_1\lambda_1\vec{u}_1 + \cdots + c_k\lambda_k\vec{u}_k = \vec{0}$ ... (1)

$\vdots$

$c_0\lambda_0^k\vec{u}_0 + c_1\lambda_1^k\vec{u}_1 + \cdots + c_k\lambda_k^k\vec{u}_k$ ... (k)

Let $f_0, f_1, \ldots, f_k$ be the Lagrange polynomials of $\lambda_0, \ldots, \lambda_k$.

That is, $f_0(\lambda_0) = 1$ and $f_0(\lambda_1) = f_0(\lambda_2) = \cdots = f_0(\lambda_k) = 0$.

Let $f_0 = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$

Then $a_0 \cdot (0) + a_1 (1) + \cdots + a_k \cdot (k)$ leads to.

$c_0 \cdot 1 \cdot \vec{u}_0 + c_1 \cdot 0 \cdot \vec{u}_1 + \cdots + c_k \cdot 0 \cdot \vec{u}_k = \vec{0}$

$\Rightarrow c_0\vec{u}_0 = \vec{0}$

Since $\vec{u}_0 \neq 0 \Rightarrow c_0 = 0$.

Similarly, we can get $c_1 = \cdots = c_k = 0$ by doing the same arguments with $f_1, \ldots, f_k$.

So β is linearly indep.

10. [extra 2pt] Let $p = 1+2x+x^2$ and $q = -4+2x+3x^2-2x^3$ be polynomials. It is known that

$$\begin{bmatrix} 1 & 0 & 0 & -4 & 0 \\ 2 & 1 & 0 & 2 & -4 \\ 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & 2 & -2 & 3 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 37 & -40 & 44 & -48 & 52 \\ -52 & 57 & -62 & 68 & -74 \\ 20 & -22 & 24 & -26 & 29 \\ 9 & -10 & 11 & -12 & 13 \\ 10 & -11 & 12 & -13 & 14 \end{bmatrix}.$$

Find polynomials $a \in \mathcal{P}_2$ and $b \in \mathcal{P}_1$ such that $ap + bq = 1$, where $\mathcal{P}_d$ is the set of all polynomials of degree at most d.

Observe that

$$\begin{array}{ccccc} 1 & x & x^2 & 1 & x \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix} 1 & & & -4 & \\ 2 & 1 & & 2 & -4 \\ 1 & 2 & 1 & 3 & 2 \\ & 1 & 2 & -2 & 3 \\ & & 1 & & -2 \end{bmatrix} \begin{bmatrix} 37 \\ -52 \\ 20 \\ 9 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow 1 \\ \leftarrow x \\ \leftarrow x^2 \\ \leftarrow x^3 \\ \leftarrow x^4 \end{array}.$$

Let $a = 37 - 52x + 20x^2$

$b = 9 + 10x$.

Then $ap + bq = 1$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	