國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第二次期中考 December 6, 2021 **Midterm 2**

姓名 Name : Solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let

$$A = \begin{bmatrix} 1 & 5 & 3 & -3 & 2 \\ 5 & 25 & 15 & -15 & 11 \\ 13 & 65 & 39 & -39 & 29 \\ 26 & 130 & 78 & -78 & 57 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 5 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is known that R is the reduced echelon form of A.

(a) [1pt] Find a basis for Row(A).

$\{(1,5,3,-3,0), (0,0,0,0,1)\}$

(b) [1pt] Find a basis for Col(A).

$\left\{\begin{pmatrix}1\\5\\13\\26\end{pmatrix}, \begin{pmatrix}2\\11\\29\\57\end{pmatrix}\right\}$.

(c) [1pt] Find a basis for ker(A).

~~$\left\{\begin{pmatrix}-5\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}3\\0\\0\\0\\1\end{pmatrix}\right\}$~~ $\left\{\begin{pmatrix}-5\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}3\\0\\0\\1\\0\end{pmatrix}\right\}$

(d) [1pt] Find the rank of A.

2

(e) [1pt] Find the nullity of A.

3

2. Let

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \right\} \text{ and } \mathbf{a} = \begin{bmatrix} -7 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) [2pt] Find a basis of V.

$V = \ker [1\ 2\ 3\ 4]$.

$\Rightarrow$ basis $= \beta_K = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\uparrow \vec{u}_1$ $\uparrow \vec{u}_2$ $\uparrow \vec{u}_3$.

(b) [1pt] Find a basis of V that contains $\mathbf{a}$.

Observe that $\vec{a} = \vec{u}_2 + \vec{u}_3$.

So $\{\vec{a}, \vec{u}_1, \vec{u}_2\}$ is a basis of A.

(c) [2pt] Let

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\}.$$

Find a basis of $U \cap V$.

$U \cap V = \ker \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{~~rref~~}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ & -1 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ & 1 & 2 & 3 \end{bmatrix}$.

$\Rightarrow$ basis $= \beta_K = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

3. Let $\mathcal{M}_{2,2}$ be the vector space of all 2×2 matrices over $\mathbb{R}$. Let

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \ A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } A_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(a) [1pt] Find a basis of $\mathcal{M}_{2,2}$.

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

(b) [1pt] Find a matrix $A \in \text{span}(\{A_1, A_2, A_3\})$ such that A is nowhere zero（每一項都不是零）.

$A_1 + A_2 + A_3 = \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix}$.

(c) [1pt] Find a matrix $A \notin \text{span}(\{A_1, A_2, A_3\})$ such that A is nowhere zero（每一項都不是零）.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ cannot be written as $c_1A_1 + c_2A_2 + c_3A_3$.

(d) [2pt] Is $\beta = \{A_1, A_2, A_3, A_4\}$ linearly independent? Explain your reasons.

No. $A_4 = A_2 - A_1$

4. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是線性獨立（linearly independent）。

請寫下線性獨立的定義，並以自己的方式、盡量白話的敘述、或是比喻來說明這個爲什麼要這樣定義？還有爲什麼要考慮這樣的概念？請給一些能幫助他人理解的例子（正面的、反面的），並提出一些這個概念的相關性質；有必要的話可以加上一些圖來輔助說明。格式沒有限制，篇幅大約半面到一面。

(If Chinese is not your native language, you may use English or the language that you prefer.)

5. [extra 2pt] Consider the vector space $\mathcal{P}_2$ of all polynomials with degree at most 2 and with real coefficients. Recall that if $p = c_0 + c_1 x + c_2 x^2$, then we define

$$\text{ptov}(p) = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.$$

Let p' and p'' be the first and the second derivative of p, respectively. Find a matrix D such that

$$\text{ptov}(p + p' + p'') = D\,\text{ptov}(p)$$

for any $p \in \mathcal{P}_2$.

Let $p = c_0 + c_1 x + c_2 x^2$. ~~ptov(p) =~~ $\Rightarrow p' = c_1 + 2c_2 x \rightarrow p'' = 2c_2$.

Then ~~p=~~ $\text{ptov}(p) = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$

$\text{ptov}(p') = \begin{bmatrix} c_1 \\ 2c_2 \\ 0 \end{bmatrix}$ $\Rightarrow \text{ptov}(p+p'+p'') = \begin{bmatrix} c_0 + c_1 + 2c_2 \\ c_1 + 2c_2 \\ c_2 \end{bmatrix}$

$\text{ptov}(p'') = \begin{bmatrix} 2c_2 \\ 0 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$

D.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	