國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第二次期中考 December 6, 2021 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let

$$A = \begin{bmatrix} 1 & 3 & -5 & -5 & 9 \\ -5 & -15 & 26 & 25 & -47 \\ 20 & 60 & -105 & -100 & 190 \\ -24 & -72 & 125 & 120 & -226 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 3 & 0 & -5 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is known that R is the reduced echelon form of A.

(a) [1pt] Find a basis for Row(A).

$$\beta_R = \left\{ (1, 3, 0, -5, -1), (0, 0, 1, 0, -2) \right\}$$

(b) [1pt] Find a basis for Col(A).

$$\beta_C = \left\{ \begin{pmatrix} 1 \\ -5 \\ 20 \\ -24 \end{pmatrix}, \begin{pmatrix} -5 \\ 26 \\ -105 \\ 125 \end{pmatrix} \right\}.$$

(c) [1pt] Find a basis for ker(A).

$$\beta_K = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(d) [1pt] Find the rank of A.

2

(e) [1pt] Find the nullity of A.

3

2. Let

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + 3x_2 + 3x_3 + 3x_4 = 0 \right\} \text{ and } \mathbf{a} = \begin{bmatrix} -6 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) [2pt] Find a basis of V.

$V = \ker \begin{bmatrix} 1 & 3 & 3 & 3 \end{bmatrix}$

$\Rightarrow$ basis $= \beta_K = \left\{ \underset{\vec{u}_1}{\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{\vec{u}_2}{\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{\vec{u}_3}{\begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\}.$

(b) [1pt] Find a basis of V that contains $\mathbf{a}$.

Since $\vec{a} = \vec{u}_2 + \vec{u}_3$,

$\{\vec{a}, \vec{u}_1, \vec{u}_2\}$ is a basis of V.

(c) [2pt] Let

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\}.$$

Find a basis of $U \cap V$.

$U \cap V = \ker \begin{bmatrix} 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

basis $= \beta_K = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

3. Let $\mathcal{M}_{2,2}$ be the vector space of all 2×2 matrices over $\mathbb{R}$. Let

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

(a) [1pt] Find a basis of $\mathcal{M}_{2,2}$.

$$\underline{\underline{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}}}$$

(b) [1pt] Find a matrix $A \in \operatorname{span}(\{A_1, A_2, A_3\})$ such that A is nowhere zero （每一項都不是零）.

$$A_1 + A_2 + A_3 = \underline{\underline{\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}}}$$

(c) [1pt] Find a matrix $A \notin \operatorname{span}(\{A_1, A_2, A_3\})$ such that A is nowhere zero （每一項都不是零）.

$\cancel{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$ $\underline{\underline{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}}$ cannot be written as $c_1A_1 + c_2A_2 + c_3A_3$

(d) [2pt] Is $\beta = \{A_1, A_2, A_3, A_4\}$ linearly independent? Explain your reasons.

$\underline{\underline{\text{No, } A_4 = A_1 - A_3}}$

4. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是線性獨立（linearly independent）。

   請寫下線性獨立的定義，並以自己的方式、盡量白話的敘述、或是比喻來說明這個爲什麼要這樣定義？還有爲什麼要考慮這樣的概念？請給一些能幫助他人理解的例子（正面的、反面的），並提出一些這個概念的相關性質；有必要的話可以加上一些圖來輔助說明。格式沒有限制，篇幅大約半面到一面。

   (If Chinese is not your native language, you may use English or the language that you prefer.)

5. [extra 2pt] Consider the vector space $\mathcal{P}_2$ of all polynomials with degree at most 2 and with real coefficients. Recall that if $p = c_0 + c_1x + c_2x^2$, then we define

$$\mathrm{ptov}(p) = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.$$

Let p' and p'' be the first and the second derivative of p, respectively. Find a matrix D such that

$$\mathrm{ptov}(p + p' + p'') = D\,\mathrm{ptov}(p)$$

for any $p \in \mathcal{P}_2$.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	