國立中山大學	NATIONAL SUN YAT-SEN UNIVERSITY
離散數學（二）	**MATH 207: Discrete Mathematics II**
期末考 June 22, 2021	**Final Exam**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

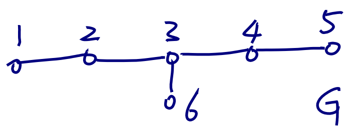
Instructions:

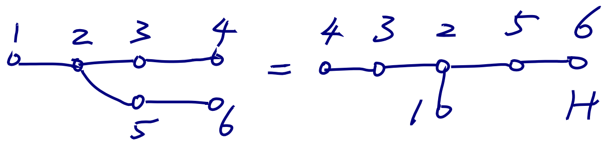
- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) [3pt] Find a permutation P such that $A = PBP^\top$.

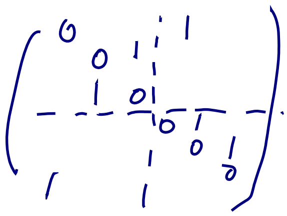
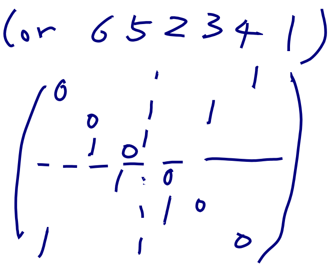




$A = A(G)$ $B = A(H)$

H is labeled by permutation $\sigma = 432561$

(or 652341)

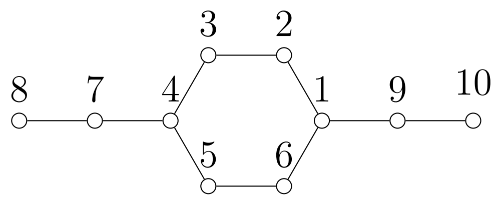
$\Rightarrow$ Choose $P =$  or 

(b) [2pt] Find the inertia (n_+, n_-, n_0) of A.

$$A \rightsquigarrow \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & 1 \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & \\ & & 1 & & & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & & & 1 \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & 1 & & & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} \Rightarrow (n_+, n_-, n_0) = \underline{\underline{(3, 3, 0)}}$$

2. Let G be the graph below and A its adjacency matrix.



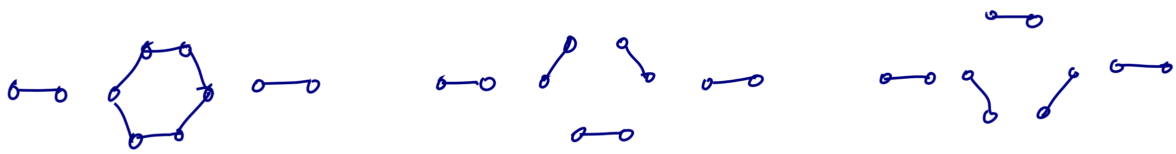
(a) [1pt] Find $\operatorname{tr}(A^2)$.

$$\operatorname{tr}(A^2) = 2\underbrace{m}_{\text{邊數}} = 2\cdot 10 = \underline{\underline{20}}$$

(b) [1pt] Find $\operatorname{tr}(A^3)$.

$$\operatorname{tr}(A^3) = 6\underbrace{t}_{\triangle\text{數}} = 6\cdot 0 = \underline{\underline{0}}$$

(c) [2pt] Draw all elementary subgraphs of G.



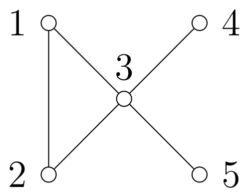
H_1 H_2 H_3

(d) [1pt] Find $\det(A)$.

	H_1	H_2	H_3
m	2	5	5
c	1	0	0
$-1^{n+m+c}\,2^c$	-2	-1	-1

$$\Rightarrow \det = -2 - 1 - 1 = \underline{\underline{-4}}$$

3. [5pt] Let G be the graph below.



Let A be the adjacency matrix of G. Find $\det(A - xI)$, the characteristic polynomial of A.

$S_0 = 1$

$S_1 = 0$

$S_2 = -m = -5$

$S_3 = 2t = 2$

$$S_4 = \underbrace{\det(\ )}_{1} + \underbrace{\det(\ )}_{1} + \underbrace{\det(\ )}_{0} + \underbrace{\det(\ )}_{0} + \underbrace{\det(\ )}_{0}$$

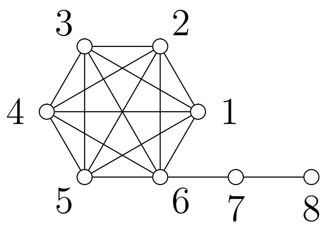
$= 2$

$S_5 = 0$ since no elementary subgraph

$$\Rightarrow \text{charpoly} = (-x)^5 + 0(-x)^4 - 5(-x)^3 + 2(-x)^2 + 2(-x) + 0$$

$$= \underline{\underline{-x^5 + 5x^3 + 2x^2 - 2x}}$$

4. [5pt] Let G be the graph below.



Find the number of spanning trees on G.

$L' = L(G)$ removing 1st row/column

$$L' = \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 5 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 5 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 5 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & & \\ -1 & 5 & -1 & -1 & -1 & & \\ -1 & -1 & 5 & -1 & -1 & & \\ -1 & -1 & -1 & 5 & -1 & & \\ -1 & -1 & -1 & -1 & 5 & -1 & \\ & & & & -1 & 1 & 0 \\ & & & & & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 5 & & & & & & \\ & 5 & & -1 & & & \\ & & 5 & & & & \\ & -1 & & 5 & & & \\ & & & & 5 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & & \\ -1 & 5 & -1 & -1 & -1 & & \\ -1 & -1 & 5 & -1 & -1 & & \\ -1 & -1 & -1 & 5 & -1 & & \\ -1 & -1 & -1 & -1 & 5 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & & \\ & 6 & & & & & \\ & & 6 & & & & \\ & & & 6 & & & \\ & & & & 6 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \Rightarrow \det = \underline{\underline{6^4}}$$

5. [extra 5pt] Let P_{n+1} be the path on $n+1$ vertices such that 1 is one of its endpoints. Let A be the adjacency matrix of P_{n+1}. Find the $1,1$-entry of A^{2n}.

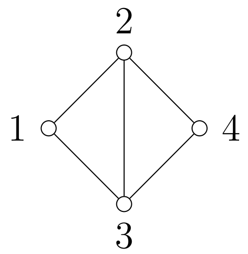
$(A^{2n})_{1,1}$ = # of walks from 1 to 1 of length $2n$

= # of ways to $\leftarrow$ and $\rightarrow$ without going to the left of 1.

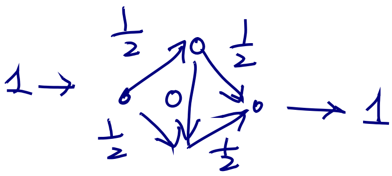
= Catalan number C_n

$= \underline{\frac{1}{n+1}\binom{2n}{n}}$

6. [extra 2pt] Let G be the graph below.

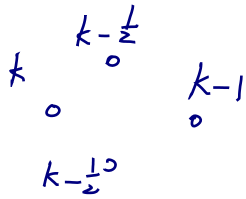


Consider G as an electronic circuit such that each edge is a wire of resistance 1Ω. Find the effective resistance from 1 to 4.



current

⇒



potential

$$\Rightarrow \text{resistance} = \frac{k-(k-1)}{1} = \underline{\underline{1}}$$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	