國立中山大學	NATIONAL SUN YAT-SEN UNIVERSITY
離散數學（二）	**MATH 207: Discrete Mathematics II**
第一次期中考	March 30, 2021 **Midterm 1**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Suppose there are 1000 balls in a box. Each ball has two numbers, X and Y, on it. The distribution of the numbers are given in the table below.

$X \backslash Y$	1	2	3	subtotal
1	50	50	200	300
2	50	50	200	300
3	100	100	200	400
subtotal	200	200	600	1000

For example, there are 200 balls whose (X, Y) is $(1, 3)$, and there are 300 balls where X is 1. Draw a ball randomly from the box, where the probability is uniformly distributed on each ball. Let X and Y be the numbers on the ball.

(a) [1pt] Find $\mathbb{E}(X)$.

(b) [1pt] Find $\mathrm{Var}(X)$.

(c) [1pt] Find $P(X = 1|Y = 1)$.

(d) [1pt] Are the random variables X and Y independent? Provide your reasons.

(e) [1pt] Find $\mathbb{E}[(X - 1)^2]$.

2. [2pt] Explain why $R(3,4) > 6$.

3. [3pt] Suppose X is a random variable such that $X \geq 5$. Show that

$$\mathbb{E}(X) \geq k \cdot P(X \geq k+5) + 5,$$

for all $k \geq 0$.

4. Let

$$M = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 \end{bmatrix}$$

be the transition matrix of a Markov chain.

(a) [1pt] Draw the digraph of M.

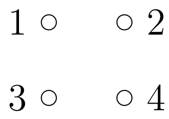
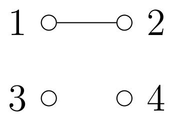
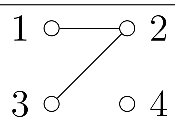
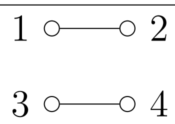
(b) [1pt] Is M irreducible? Provide your reasons.

(c) [2pt] If M has a unique stationary state, find it; otherwise, find two different stationary states.

(d) [1pt] Does $\lim_{k\to\infty} M^k$ exist? Provide your reasons.

5. [5pt] Let $G(n, p)$ be the Erdős–Rényi random graph, where n is the number of vertices and p is the probability for each edge to occur. Let X be the number of isolated vertices on $G(n, p)$. Find $\mathbb{E}(X)$. (You may provide the answer for small n to get some partial credits.)

   An isolated vertex is a vertex that is not adjacent to any other vertex. The table below provides some examples.

$G(n, p)$	isolated vertices	X
	$1, 2, 3, 4$	4
	$3, 4$	2
	4	1
	$\emptyset$	0

6. [extra 2pt] Randomly put the four numbers $1, 2, 3, 4$ in a line such that the probability to get each permutation is $\frac{1}{4!}$. A *right-to-left maximum* is a number that is greater than all numbers to its right. For example, 3412 has only two right-to-left maxima 2 and 4; 4231 has three right-to-left maxima 1, 3, and 4; and all four numbers in 4321 are right-to-left maxima. Let X be the number of right-to-left maxima. Find $\mathbb{E}(X)$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	