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| 國立中山大學 | NATIONAL SUN YAT-SEN UNIVERSITY |
| 離散數學（二） | **MATH 207: Discrete Mathematics II** |
| 第一次期中考 | March 30, 2021 **Midterm 1** |

姓名 Name ： solution

學號 Student ID # ：

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| Lecturer: | Jephian Lin 林晉宏 |
| Contents: | cover page, |
| | **5 pages** of questions, |
| | score page at the end |
| To be answered: | on the test paper |
| Duration: | **110 minutes** |
| Total points: | **20 points** + 2 extra points |

**Do not open this packet until instructed to do so.**

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Suppose there are 1000 balls in a box. Each ball has two numbers, $X$ and $Y$, on it. The distribution of the numbers are given in the table below.

| $X \backslash Y$ | 1 | 2 | 3 | subtotal |
|---|---|---|---|---|
| 1 | 50 | 50 | 200 | 300 |
| 2 | 50 | 50 | 200 | 300 |
| 3 | 100 | 100 | 200 | 400 |
| subtotal | 200 | 200 | 600 | 1000 |

For example, there are 200 balls whose $(X, Y)$ is $(1, 3)$, and there are 300 balls where $X$ is 1. Draw a ball randomly from the box, where the probability is uniformly distributed on each ball. Let $X$ and $Y$ be the numbers on the ball.

(a) [1pt] Find $\mathbb{E}(X)$.

$1 \cdot 0.3 + 2 \cdot 0.3 + 3 \cdot 0.4 = 0.3 + 0.6 + 1.2 = \underline{2.1}$

(b) [1pt] Find $\mathrm{Var}(X)$.

$\mathbb{E}(X^2) = 1 \cdot 0.3 + 4 \cdot 0.3 + 9 \cdot 0.4 = 0.3 + 1.2 + 3.6 = 5.1$

$\mathbb{E}(X)^2 = 2.1^2 = 4.41$

$\mathrm{Var}(X) = 5.1 - 4.41 = \underline{0.69}$

(c) [1pt] Find $P(X = 1|Y = 1)$.

$50/200 = \underline{\frac{1}{4}}$ or $\underline{0.25}$

(d) [1pt] Are the random variables $X$ and $Y$ independent? Provide your reasons.

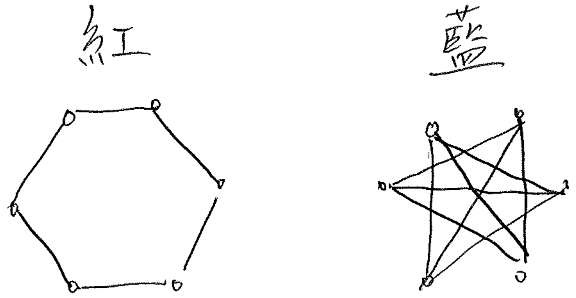
<u>No.</u> For ~~ea~~ example, $P(X=1) = 0.3$ but $P(X=1|Y=1) = 0.25$.

(e) [1pt] Find $\mathbb{E}[(X - 1)^2]$.

$\mathbb{E}[(X-1)^2] = \mathbb{E}(X^2) - 2\mathbb{E}(X) + \mathbb{E}(1)$

$= 5.1 - 2 \cdot 2.1 + 1 = 5.1 - 4.2 + 1 = \underline{1.9}$.

2. [2pt] Explain why $R(3,4) > 6$.




This edge 2-coloring of $K_6$
has no red $K_3$ and no blue $K_4$

$\Rightarrow R(3,4) > 6$.

3. [3pt] Suppose $X$ is a random variable such that $X \geq 5$. Show that

$$\mathbb{E}(X) \geq k \cdot P(X \geq k+5) + 5,$$

for all $k \geq 0$.

Let $X' = X - 5$. Then $X' \geq 0$.

By Markov's inequality,

$$\mathbb{E}(X') \geq k \cdot P(X' \geq k)$$

$$\Rightarrow \mathbb{E}(X) - 5 \geq k \cdot P(X - 5 \geq k)$$

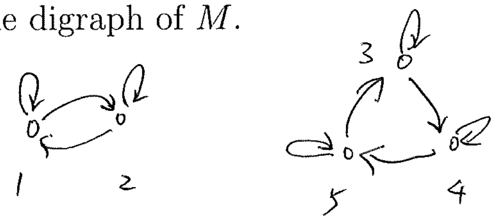
$$\Rightarrow \mathbb{E}(X) \geq k \cdot P(X \geq k+5) + 5.$$

4. Let

$$M = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 \end{bmatrix} \end{array}$$

be the transition matrix of a Markov chain.

(a) [1pt] Draw the digraph of $M$.




(b) [1pt] Is $M$ irreducible? Provide your reasons.

No, since the digraph is not ~~strongly~~ connected. (e.g., 1 cannot go to 3).

(c) [2pt] If $M$ has a unique stationary state, find it; otherwise, find two different stationary states.

$$M - I = \begin{pmatrix} -0.5 & 0.5 & & & \\ 0.5 & -0.5 & & & \\ & & -0.5 & 0.5 & \\ & & & -0.5 & 0.5 \\ & & 0.5 & & -0.5 \end{pmatrix}$$ has left kernel spanned by

$$\left\{ \underline{\left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0\right)},\ \underline{\left(0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)} \right\}.$$

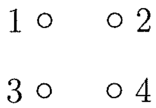
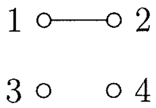
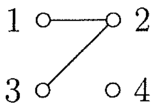
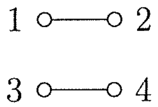
(d) [1pt] ~~Suppose $\lim_{k\to\infty} M^k$ exists. Find the limit.~~ Does $\lim\limits_{k\to\infty} M^k$ exist?

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{5} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{5} \end{pmatrix}$$

Yes, since both components of M are primitive (have loop).

5. [5pt] Let $G(n,p)$ be the Erdős–Rényi random graph, where $n$ is the number of vertices and $p$ is the probability for each edge to occur. Let $X$ be the number of isolated vertices on $G(n,p)$. Find $\mathbb{E}(X)$. (You may provide the answer for small $n$ to get some partial credits.)

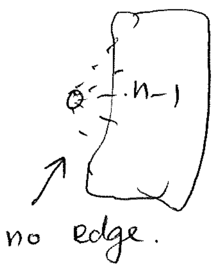
An isolated vertex is a vertex that is not adjacent to any other vertex. The table below provides some examples.

| $G(n,p)$ | isolated vertices | $X$ |
|---|---|---|
|  | 1, 2, 3, 4 | 4 |
|  | 3, 4 | 2 |
|  | 4 | 1 |
|  | $\emptyset$ | 0 |

$$X_i = \begin{cases} 1 & \text{if vertex } i \text{ is isolated.} \\ 0 & \text{o.w.} \end{cases}$$

$P(X_i = 1) = (1-p)^{n-1}$.

$\mathbb{E}(X_i) = (1-p)^{n-1}$ for each $i$.




Then $X = X_1 + \dots + X_n$

$\mathbb{E}(X) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n) = \underline{n \cdot (1-p)^{n-1}}$

For $n=1$, $\mathbb{E}(X) = (1-p)^0 = 1$.

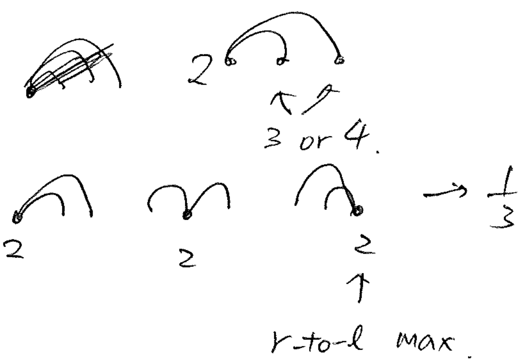
$n=2$, $= 2\cdot(1-p)^{1} = 2(1-p)$

⋮

6. [extra 2pt] Randomly put the four numbers $1, 2, 3, 4$ in a line such that the probability to get each permutation is $\frac{1}{4!}$. A *right-to-left maximum* is a number that is greater than all numbers to its right. For example, 3412 has only two right-to-left maxima 2 and 4; 4231 has three right-to-left maxima 1, 3, and 4; and all four numbers in 4321 are right-to-left maxima. Let $X$ be the number of right-to-left maxima. Find $\mathbb{E}(X)$.

Let $X_i = \begin{cases} 1 & \text{if } i \text{ is a right-to-left max} \\ 0 & \text{o.w.} \end{cases}$

Then $P(X_i = 1) = \frac{1}{5-i}$.




So $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3 + X_4)$

$$= \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1 = \frac{25}{12}.$$

[END]

| Page | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 2 | |
| Total | 20 (+2) | |