國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

離散數學（二） **MATH 207: Discrete Mathematics II**

第一次期中考 March 30, 2021 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Suppose there are 1000 balls in a box. Each ball has two numbers, X and Y, on it. The distribution of the numbers are given in the table below.

$X\backslash Y$	1	2	3	subtotal
1	50	50	200	300
2	50	50	200	300
3	100	100	200	400
subtotal	200	200	600	1000

For example, there are 200 balls whose (X, Y) is $(1, 3)$, and there are ~~300~~ 200 balls where ~~X~~ Y is 1. Draw a ball randomly from the box, where the probability is uniformly distributed on each ball. Let X and Y be the numbers on the ball.

(a) [1pt] Find $\mathbb{E}(Y)$.

$1\cdot 0.2 + 2\cdot 0.2 + 3\cdot 0.6 = 0.2 + 0.4 + 1.8 = \underline{2.4}$.

(b) [1pt] Find $\mathrm{Var}(Y)$.

$\mathbb{E}(Y^2) = 1\cdot 0.2 + 4\cdot 0.2 + 9\cdot 0.6 = 0.2 + 0.8 + 5.4 = 6.4$

$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = 6.4 - 2.4^2 = 6.4 - 5.76 = \underline{0.64}$

(c) [1pt] Find $P(Y = 1|X = 1)$.

$50/300 = \underline{\frac{1}{6}}$

(d) [1pt] Are the random variables X and Y independent? Provide your reasons.

$\underline{\text{No}}$, e.g., $P(Y=1) = 0.2$

but $P(Y=1|X=1) = \frac{1}{6}$.

(e) [1pt] Find $\mathbb{E}[(Y-1)^2]$.

$\mathbb{E}[(Y-1)^2] = \mathbb{E}(Y^2) - 2\mathbb{E}(Y) + \mathbb{E}(1)$

$= 6.4 - 2\cdot 2.4 + 1$

$= 6.4 - 4.8 + 1 = \underline{2.6}$

2. [2pt] Explain why $R(3,4) > 6$.

See ver. A.

3. [3pt] Suppose X is a random variable such that $X \geq 7$. Show that

$$\mathbb{E}(X) \geq k \cdot P(X \geq k+7) + 7,$$

for all $k \geq 0$.

Let $X' = X - 7$. Then $X' \geq 0$.

By Markov's inequality,

$$\mathbb{E}(X') \geq k \cdot P(X' \geq k)$$

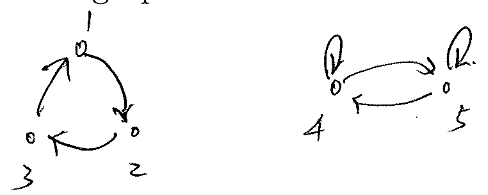
$$\Rightarrow \mathbb{E}(X) \geq k \cdot P(X \geq k+7) + 7.$$

4. Let

$$M = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

be the transition matrix of a Markov chain.

(a) [1pt] Draw the digraph of M.



(b) [1pt] Is M irreducible? Provide your reasons.

No, digraph is not strongly connected.

e.g., 1 cannot go to 4.

(c) [2pt] If M has a unique stationary state, find it; otherwise, find two different stationary states.

M−I has its left kernel spanned by

$$\left\{ \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0\right), \cancel{\left(\tfrac{1}{2}, \tfrac{1}{2}\right.} \left(0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}\right) \right\}.$$

(d) [1pt] ~~Suppose $\lim_{k\to\infty} M^k$ exists. Find the limit.~~ Does $\lim_{k\to\infty} M^k$ exist?

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

No, since

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

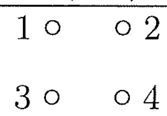
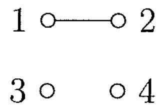
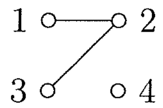
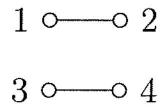
$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

are periodic.

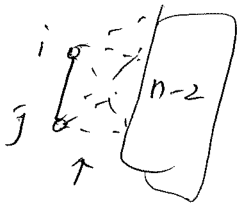
5. [5pt] Let $G(n,p)$ be the Erdős–Rényi random graph, where n is the number of vertices and p is the probability for each edge to occur. Let X be the number of isolated edges on $G(n,p)$. Find $\mathbb{E}(X)$. (You may provide the answer for small n to get some partial credits.)

An isolated edge is an edge that does not share an endpoint with any other edge. The table below provides some examples.

$G(n,p)$	isolated edges	X
	$\emptyset$	0
	12	1
	$\emptyset$	0
	12, 34	2

For $i<j$,

Let $X_{ij} = \begin{cases} 1 & \text{if } ij \text{ is an } \cancel{\text{edge}} \text{ isolated edge.} \\ 0 \end{cases}$



$2\cdot(n-2)$ non-edges.

$$P(X_{ij}=1) = p\cdot(1-p)^{2\cdot(n-2)}$$

$$\Rightarrow \mathbb{E}(X) = \mathbb{E}\Big(\sum_{i<j} X_{ij}\Big)$$

$$= \sum_{i<j} \mathbb{E}(X_{ij})$$

$$= \binom{n}{2}\cdot p\cdot(1-p)^{2\cdot(n-2)}.$$

For $n=2$, $\mathbb{E}(X) = 1\cdot p = p$

$=3$, $= 3\cdot p\cdot(1-p)^2$

6. [extra 2pt] Randomly put the four numbers $1, 2, 3, 4$ in a line such that the probability to get each permutation is $\frac{1}{4!}$. A *right-to-left maximum* is a number that is greater than all numbers to its right. For example, 3412 has only two right-to-left maxima 2 and 4; 4231 has three right-to-left maxima 1, 3, and 4; and all four numbers in 4321 are right-to-left maxima. Let X be the number of right-to-left maxima. Find $\mathbb{E}(X)$.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	