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國立中山大學　　NATIONAL SUN YAT-SEN UNIVERSITY

離散數學（二）　　**MATH 207: Discrete Mathematics II**

第二次期中考　　May 11, 2021　　**Midterm 2**

姓名 Name : solution

學號 Student ID # : ______________

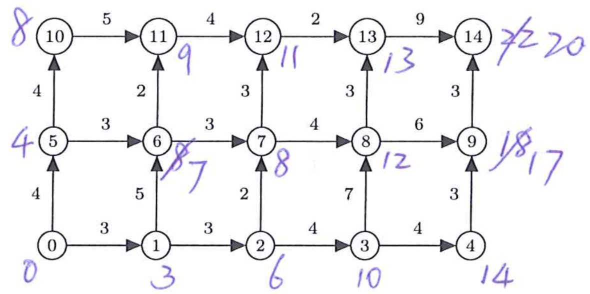
Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let Γ be the directed graph as shown below. The number on each edge is its weight (distance).



(a) [1pt] What is the distance from 0 to 10?

8

(b) [1pt] What is the distance from 0 to 11?

9

(c) [1pt] What is the distance from 0 to 12?

11

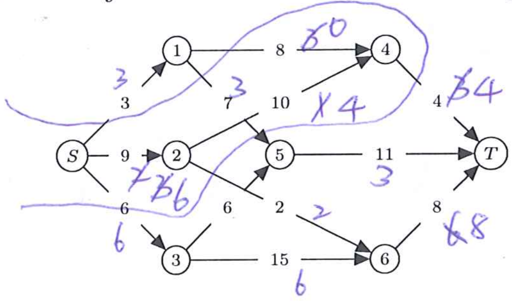
(d) [1pt] What is the distance from 0 to 13?

13

(e) [1pt] What is the distance from 0 to 14?

20

2. [5pt] Let Γ be the directed graph below, where S and T are the source and the sink, respectively. The number on each edge is its capacity.



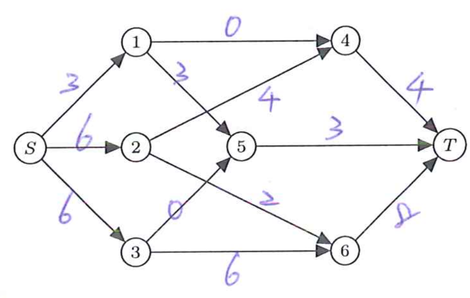
Find a flow function f with the maximum value and a cut (A, B) with the minimum capacity.

[Note: Use the graph at the bottom to answer your flow on each edge and your A and B.]

max value of flow

= min capacity of cut

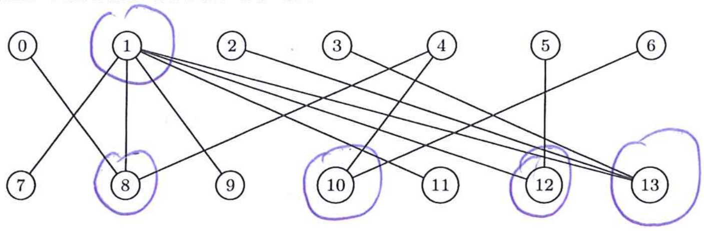
= 15.

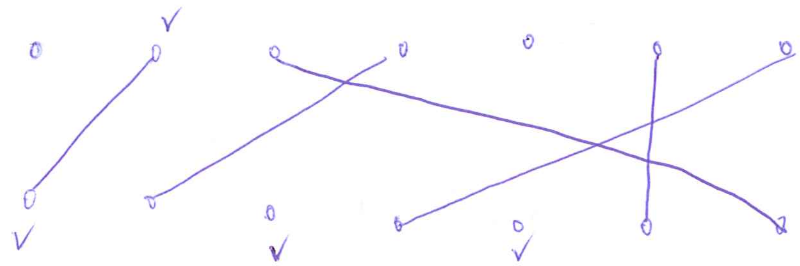


$A =$ {S, 2, 4}.

$B =$ {1, 3, 5, 6, T}.

3. [5pt] Let G be the bipartite graph below. Find a maximum matching and a minimum vertex cover of G.





max matching = {17, 48, 3 13, 5 12, 6 10}.

min vtx cover = {1, 8, 10, 12, 13}.

4. [5pt] Let (Γ, S, T, c) be a network on the directed graph Γ, where S, T, and c are the source, the sink, and the capacity function, respectively. Let f be a flow on this network and (A, B) a cut. Show that

$$\sum_{\substack{(u,v)\in E(\Gamma)\\ u\in A\\ v\in B}} f(u,v) - \sum_{\substack{(u,v)\in E(\Gamma)\\ u\in B\\ v\in A}} f(u,v) = \sum_{\substack{v\in V(\Gamma)\\ (S,v)\in E(\Gamma)}} f(S,v) - \sum_{\substack{u\in V(\Gamma)\\ (u,S)\in E(\Gamma)}} f(u,S).$$

By the conservation law.

$$\sum_{v: x\to v} f(x,v) - \sum_{u: u\to x} f(u,x) = 0 \text{ for all } x\in A\setminus\{S\}.$$

So

$$\text{RHS} = \sum_{v: S\to v} f(S,v) - \sum_{u: u\to S} f(u,S)$$

$$= \sum_{x\in A} \left[\sum_{v: x\to v} f(x,v) - \sum_{u: u\to x} f(u,x)\right]$$

$$= \sum_{\substack{u,v\\ (u,v)\in E(\Gamma)\\ u,v\in A}} \left[f(u,v) - f(u,v)\right] + \sum_{\substack{u,v\\ (u,v)\in E(\Gamma)\\ u\in A,\, v\in B}} f(u,v) - \sum_{\substack{u,v\\ (u,v)\in E(\Gamma)\\ u\in B,\, v\in A}} f(u,v)$$

$$= \text{LHS}.$$

5. [extra 2pt] Let $\mathbf{v}_1, \ldots, \mathbf{v}_5$ be the columns of

$$\begin{array}{c}\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 2 \end{bmatrix} \\ \begin{matrix} \;v_1 & v_4 & v_5 & v_2 & v_3 \end{matrix}\end{array}.$$

Let $\mathbf{e}_1 = (1, 0, 0)^\top$. Find a subset $S \subseteq \{1, 2, 3, 4, 5\}$ such that $\{\mathbf{v}_i\}_{i \in S}$ is ~~linearly independent~~ a basis and

$$\sum_{i \in S} \langle \mathbf{e}_1, \mathbf{v}_i \rangle$$

is minimized. Here $\langle \mathbf{x}, \mathbf{y} \rangle$ is the standard inner product in $\mathbb{R}^3$.

$E = \{\vec{v}_1, \ldots, \vec{v}_5\} \Rightarrow (E, \{\text{indep subsets of } E\})$ is a matroid.

$\Rightarrow$ apply the greedy algorithm.

weight $w(v_i) = \langle \vec{e}_1, \vec{v}_i \rangle$

	v_1	v_4	v_5	v_2	v_3	sorted
weight	1	2	3	4	5.	
choose or not	✓	X	✓	X	✓	

$\Rightarrow$ $\underline{S = \{\vec{v}_1, \vec{v}_3, \vec{v}_5\}}$ minimized $\sum \langle e_1, v_i \rangle$

as $\underline{1+3+5 = 9}$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	

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