國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

期末考 December 19, 2022 **Final Exam**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using a calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Let $X = \{p, q, r\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **injective but not surjective**.

2. [1pt] Let $X = \{p, q, r\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **surjective but not injective**.

3. [1pt] Let $X = \{p, q, r\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **bijective**.

4. [1pt] Define a function $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that f is **linear**.

5. [1pt] Define a function $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that f is **not linear**.

6. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \text{ and}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear function such that $f(\mathbf{u}_i) = \mathbf{v}_i$ for $i = 1, 2, 3$.

(a) [1pt] Find $f(\mathbf{u}_1 + \mathbf{u}_2)$.

(b) [2pt] Find $f(\mathbf{e}_2)$ and $f(\mathbf{e}_3)$.

(c) [1pt] Find the matrix representation $[f]$ such that $[f]\mathbf{u} = f(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^3$.

(d) [1pt] Find $\operatorname{rank}(f)$ and $\operatorname{null}(f)$.

7. Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Let $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis of $\mathbb{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}, \text{ and } \mathbf{u}_3 = \begin{bmatrix} -1 \\ 3 \\ -6 \end{bmatrix}.$$

(a) [2pt] Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that

$$[\mathbf{v}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Find $\mathbf{v}$ and $[\mathbf{w}]_\beta$.

(b) [3pt] Find the change of basis matrices $[\mathrm{id}]_\alpha^\beta$ and $[\mathrm{id}]_\beta^\alpha$.

8. Let V be a vector space and $\beta = \{\mathbf{u}_1, \dots, \mathbf{u}_d\}$ a basis of V. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of the *vector representation* as clear as possible.

   (a) [2pt] Suppose $\mathbf{v} \in V$ is a vector. Define what is the vector representation $[\mathbf{v}]_\beta$ with respect to β and use a few sentences to explain the definition.

   (b) [1pt] What might happen if β is not a basis?

   (c) [2pt] Provide an example of $[\mathbf{v}]_\beta$ with $V = \mathbb{R}^2$ and an example of $[\mathbf{v}]_\beta$ with $V = \mathcal{P}_1$, the space of all real polynomials of degree at most 1.

9. [extra 5pt] Let $\mathcal{M}_{2,3}$ be the space of all 2×3 real matrices. Let

$$\beta = \{E_{1,1}, E_{1,2}, E_{1,3}, E_{2,1}, E_{2,2}, E_{2,3}\}$$

be the standard basis of $\mathcal{M}_{2,3}$, where

$$E_{1,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_{2,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, E_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the linear function $f : \mathcal{M}_{2,3} \to \mathcal{M}_{2,3}$ defined by $f(X) = AX$ with

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find the matrix representation $[f]_\beta^\beta$.

10. [extra 2pt] Let $\mathcal{P}_3$ be the space of all real polynomials of degree at most 3. Let

$$\beta = \{f_1(x), f_2(x), f_3(x), f_4(x)\}$$

be a basis of $\mathcal{P}_3$, where

$$\begin{aligned} f_1(x) &= \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}, \\ f_2(x) &= \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}, \\ f_3(x) &= \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}, \\ f_4(x) &= \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}. \end{aligned}$$

Let $p(x) = 1 - 2x + 3x^2 - 4x^3$. Find the vector representation $[p(x)]_\beta$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	(+5)	
6	(+2)	
Total	20 (+7)	