國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

期末考 December 19, 2022 **Final Exam**

姓名 Name : solution

學號 Student ID # :

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using a calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Let $X = \{p, q, r\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **injective but not surjective**.

$Y = \{1,2,3,4\}$

$$f: \begin{array}{l} p \mapsto 1 \\ q \mapsto 2 \\ r \mapsto 3 \\ \quad\;\; 4 \end{array}$$

2. [1pt] Let $X = \{p, q, r\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **surjective but not injective**.

$Y = \{1\}$.

$$f: \begin{array}{l} p \mapsto 1 \\ q \mapsto 1 \\ r \mapsto 1 \end{array}$$

3. [1pt] Let $X = \{p, q, r\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **bijective**.

$Y = \{1,2,3\}$

$$f: \begin{array}{l} p \mapsto 1 \\ q \mapsto 2 \\ r \mapsto 3 \end{array}$$

4. [1pt] Define a function $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that f is **linear**.

Define $f(x,y,z) = (0,0)$ for all $(x,y,z) \in \mathbb{R}^3$.

5. [1pt] Define a function $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that f is **not linear**.

Define $f(x,y,z) = (1,1)$ for all $(x,y,z) \in \mathbb{R}^3$.

6. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Let

$$\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}3\\-2\\1\end{bmatrix}, \text{ and}$$

$$\mathbf{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}3\\1\end{bmatrix}.$$

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear function such that $f(\mathbf{u}_i) = \mathbf{v}_i$ for $i = 1, 2, 3$.

(a) [1pt] Find $f(\mathbf{u}_1 + \mathbf{u}_2)$.

$f(\vec{u}_1+\vec{u}_2) = f(\vec{u}_1)+f(u_2) = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix}2\\4\end{bmatrix}$.

(b) [2pt] Find $f(\mathbf{e}_2)$ and $f(\mathbf{e}_3)$.

$\vec{e}_2 = \vec{u}_2 + 2\vec{u}_1$

So $f(\vec{e}_2) = f(\vec{u}_2) + 2f(\vec{u}_1) = \vec{v}_2 + 2\vec{v}_1 = \cancel{\begin{bmatrix}7\\5\end{bmatrix}}$ $\begin{bmatrix}3\\7\end{bmatrix}$

$\vec{e}_3 = \vec{u}_3 + 2\vec{u}_2 + \vec{u}_1$

So $f(\vec{e}_3) = f(\vec{u}_3) + 2f(\vec{u}_2) + f(\vec{u}_1) = \vec{v}_3 + 2\vec{v}_2 + \vec{v}_1 = \begin{bmatrix}6\\6\end{bmatrix}$.

(c) [1pt] Find the matrix representation $[f]$ such that $[f]\mathbf{u} = f(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^3$.

Note: $f(\vec{e}_1) = f(\vec{u}_1) = \begin{bmatrix}1\\3\end{bmatrix}$. So $[f] = \begin{bmatrix} | & | & | \\ f(\vec{e}_1) & f(\vec{e}_2) & f(\vec{e}_3) \\ | & | & | \end{bmatrix}$

$= \begin{bmatrix} 1 & \cancel{4}3 & 6 \\ 3 & \cancel{5}7 & 6 \end{bmatrix}$,

(d) [1pt] Find rank(f) and null(f).

$[f] \rightsquigarrow \begin{bmatrix} 1 & 3 & 6 \\ 0 & -2 & -12 \end{bmatrix}$. # of pivots $= 2$.

rank$([f])$ = # of pivots $= 2$

null$([f]) = 3 -$ rank$([f]) = 1$.

7. Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Let $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis of $\mathbb{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}, \text{ and } \mathbf{u}_3 = \begin{bmatrix} -1 \\ 3 \\ -6 \end{bmatrix}.$$

(a) [2pt] Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that

$$[\mathbf{v}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Find $\mathbf{v}$ and $[\mathbf{w}]_\beta$.

Since $[\vec{v}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = 1\vec{u}_1 + 2\vec{u}_2 + 3\vec{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

Then solve $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{w}$ by

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -2 & -3 & 3 & 2 \\ 5 & 8 & -6 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & -2 & -1 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\Rightarrow \begin{cases} c_1 = 11. \\ c_2 = -2 \\ c_3 = 6 \end{cases} \qquad \text{So } [\vec{w}]_\beta = \begin{bmatrix} 11 \\ -2 \\ 6 \end{bmatrix}.$$

(b) [3pt] Find the change of basis matrices $[\mathrm{id}]_\alpha^\beta$ and $[\mathrm{id}]_\beta^\alpha$.

$$[id]_\beta^\alpha = \begin{bmatrix} | & | & | \\ [\vec{u}_1]_\alpha & [\vec{u}_2]_\alpha & [\vec{u}_3]_\alpha \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & 3 \\ 5 & 8 & -6 \end{bmatrix}.$$

Thus, $[id]_\alpha^\beta = \left([id]_\beta^\alpha\right)^{-1} = \begin{bmatrix} -6 & 4 & 3 \\ 3 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$ since

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & -3 & 3 & 0 & 1 & 0 \\ 5 & 8 & -6 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & -2 & -1 & -5 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & 3 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right].$$

8. Let V be a vector space and $\beta = \{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ a basis of V. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of the *vector representation* as clear as possible.

   (a) [2pt] Suppose $\mathbf{v} \in V$ is a vector. Define what is the vector representation $[\mathbf{v}]_\beta$ with respect to β and use a few sentences to explain the definition.

   (b) [1pt] What might happen if β is not a basis?

   (c) [2pt] Provide an example of $[\mathbf{v}]_\beta$ with $V = \mathbb{R}^2$ and an example of $[\mathbf{v}]_\beta$ with $V = \mathcal{P}_1$, the space of all real polynomials of degree at most 1.

9. [extra 5pt] Let $\mathcal{M}_{2,3}$ be the space of all 2×3 real matrices. Let

$$\beta = \{E_{1,1}, E_{1,2}, E_{1,3}, E_{2,1}, E_{2,2}, E_{2,3}\}$$

be the standard basis of $\mathcal{M}_{2,3}$, where

$$E_{1,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_{2,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, E_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the linear function $f : \mathcal{M}_{2,3} \to \mathcal{M}_{2,3}$ defined by $f(X) = AX$ with

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find the matrix representation $[f]_{\beta}^{\beta}$.

$$f(E_{1,1}) = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \xrightarrow{\cancel{E_{1,1}}} [f(E_{1,1})]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$$f(E_{1,2}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \longrightarrow [f(E_{1,2})]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

$$f(E_{1,3}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \longrightarrow [f(E_{1,3})]_\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

$$f(E_{2,1}) = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \longrightarrow [f(E_{2,1})]_\beta = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$f(E_{2,2}) = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to [f(E_{2,2})]_\beta = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$f(E_{2,3}) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow [f(E_{2,3})]_\beta = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

$$[f]_\beta^\beta = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ \hline 3 & 0 & 0 & 4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 & 0 & 4 \end{array}\right]$$

10. [extra 2pt] Let $\mathcal{P}_3$ be the space of all real polynomials of degree at most 3. Let

$$\beta = \{f_1(x), f_2(x), f_3(x), f_4(x)\}$$

be a basis of $\mathcal{P}_3$, where

$$f_1(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)},$$
$$f_2(x) = \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)},$$
$$f_3(x) = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)},$$
$$f_4(x) = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}.$$

Let $p(x) = 1 - 2x + 3x^2 - 4x^3$. Find the vector representation $[p(x)]_\beta$.

Suppose $p(x) = d_1 f_1(x) + d_2 f_2(x) + d_3 f_3(x) + d_4 f_4(x)$.

Then $\begin{cases} d_1 = p(1) = -2 \\ d_2 = p(2) = -23 \\ d_3 = p(3) = -86 \\ d_4 = p(4) = -215 \end{cases}$.

So $[p(x)]_\beta = \begin{bmatrix} -2 \\ -23 \\ -86 \\ -215 \end{bmatrix}$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	(+5)	
6	(+2)	
Total	20 (+7)	