國立中山大學	NATIONAL SUN YAT-SEN UNIVERSITY
線性代數（一）	**MATH 103A / GEAI 1215A: Linear Algebra I**
第一次期中考	October 3, 2022 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 5 \\ -1 \\ -3 \\ 1 \end{bmatrix}.$$

(a) [1pt] Find the length of $\mathbf{u}$.

$\|u\| = \sqrt{1^2+1^2+(-1)^2+(-1)^2} = \underline{2}.$

(b) [1pt] Let θ be the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Find $\cos\theta$.

$\|v\| = \sqrt{5^2+(-1)^2+(-3)^2+1^2} = \sqrt{25+1+9+1} = 6.$

$\langle u,v\rangle = 1\cdot 5 + 1\cdot(-1) + (-1)(-3) + (-1)\cdot 1 = 5-1+3-1 = 6$

$\cos\theta = \dfrac{\langle u,v\rangle}{\|u\|\cdot\|v\|} = \dfrac{6}{2\cdot 6} = \underline{\dfrac{1}{2}}.$

(c) [1pt] Find a vector in span($\{\mathbf{u}, \mathbf{v}\}$). Provide your reasons.

For example, $1\vec{u} + 1\vec{v} = \underline{\begin{bmatrix} 6 \\ 0 \\ -4 \\ 0 \end{bmatrix}}$ is in span $(\{\vec{u},\vec{v}\})$.

(d) [1pt] Find a vector **NOT** in span($\{\mathbf{u}, \mathbf{v}\}$). Provide your reasons.

For example, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in span $(\{\vec{u},\vec{v}\})$

Since $\begin{bmatrix} 1 & 5 \\ 1 & -1 \\ -1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underline{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$ has no solution.

(e) [1pt] Find a nonzero vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

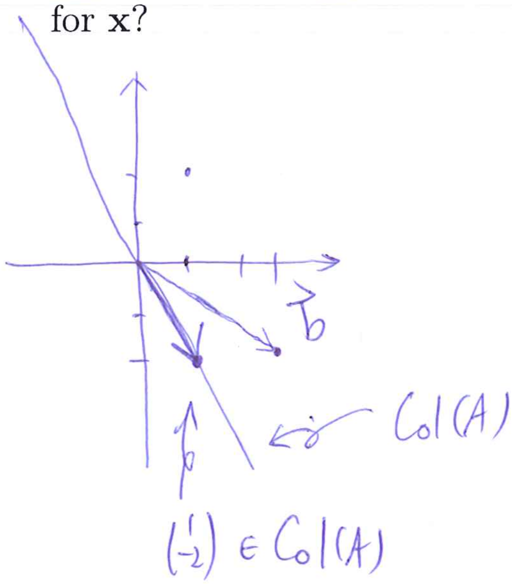
Solve $\begin{cases} x+y-z-w=0 \\ 5x-y-3z+w=0 \end{cases} \rightsquigarrow \begin{cases} x+y-z-w=0 \\ -6y+2z+6w=0 \end{cases}$

For example, $\underline{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$ is a solution.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

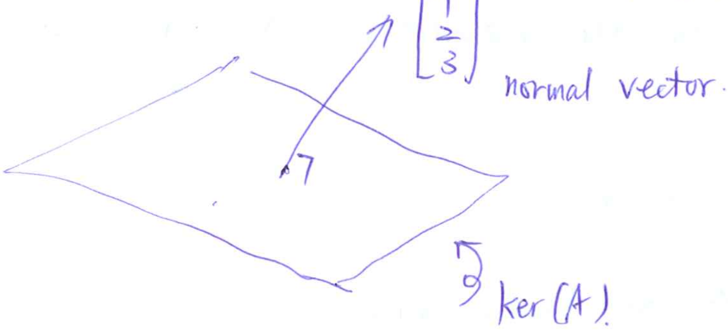
(a) [2pt] Draw the vector $\mathbf{b}$ and the subspace Col(A) in $\mathbb{R}^2$. Mark at least one vector in Col(A) in your drawing. Does $A\mathbf{x} = \mathbf{b}$ have a solution for $\mathbf{x}$?



No solution since $\vec{b} \notin$ Col(A).

(b) [1pt] Draw the subspace ker(A) and its normal vector in $\mathbb{R}^3$.

$\ker(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x+2y+3z=0 \right\}$



(c) [2pt] Find $\mathbf{h}_1$ and $\mathbf{h}_2$ such that $\ker(A) = \text{span}(\{\mathbf{h}_1, \mathbf{h}_2\})$.

Solve. $x+2y+3z=0$

Set $y=1, z=0 \Rightarrow \vec{h}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Set $y=0, z=1 \Rightarrow \vec{h}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 2 & 4 & 7 & 15 \\ -1 & -2 & -2 & -6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -7 \\ -17 \\ 4 \end{bmatrix}.$$

(a) [2pt] Find the reduced echelon form of the augmented matrix $[\, A \,|\, \mathbf{b} \,]$.

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 7 & -7 \\ 2 & 4 & 7 & 15 & -17 \\ -1 & -2 & -2 & -6 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 7 & -7 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 1 & -3 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rref.

(b) [3pt] Find $\mathbf{p}$, $\mathbf{h}_1$, and $\mathbf{h}_2$ such that the set of general solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\{\mathbf{p} + c_1\mathbf{h}_1 + c_2\mathbf{h}_2 : c_1, c_2 \in \mathbb{R}\}.$$

The rref leads to the equation $\begin{cases} x_1 + 2x_2 + 4x_4 = 2 \\ x_3 + x_4 = -3 \end{cases} \cdots (*)$

Free vars $= x_2, x_4$.

Set $x_2 = x_4 = 0$ and solve $(*)$:

$$\vec{p} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}.$$

Homogeneous equations $\begin{cases} x_1 + 2x_2 + 4x_4 = 0 \\ x_3 + x_4 = 0 \end{cases} \cdots (**)$

Set $x_2 = 1, x_4 = 0$ and solve $(**)$: $\vec{h}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Set $x_2 = 0, x_4 = 1$ and solve $(**)$: $\vec{h}_2 = \begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

4. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of a *subspace* as clear as possible.

   (a) [3pt] Define what is a subspace and use a few sentences to explain the definition.

   (b) [2pt] Provide an example of a subspace and an example of a set that is not a subspace. Provide your reasons.

5. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ t \end{bmatrix}.$$

Find t such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 5 \\ 1 & 1 & 3 & 3 & t \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & t-3 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & t-7 \end{array}\right].$$

$A\vec{x} = \vec{b}$ is consistent if and only if $\underline{\underline{t=7.}}$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	