國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） MATH 103A / GEAI 1215A: Linear Algebra I

第一次期中考 October 3, 2022 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let

$$\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) [1pt] Find the length of $\mathbf{u}$.

$\|u\| = \sqrt{5^2+1^2+3^2+1^2} = 6.$

(b) [1pt] Let θ be the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Find $\cos\theta$.

$\|v\| = \sqrt{1^2+1^2+1^2+1^2} = 2.$

$\langle u, v \rangle = 5\cdot 1 + 1\cdot 1 + 3\cdot 1 + 1\cdot 1 = 10$

$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\cdot\|v\|} = \frac{10}{6\cdot 2} = \frac{5}{6}$

(c) [1pt] Find a vector in span($\{\mathbf{u}, \mathbf{v}\}$). Provide your reasons.

For example, $1\vec{u} + 1\vec{v} = \begin{bmatrix} 6 \\ 2 \\ 4 \\ 2 \end{bmatrix}$ is in span$(\{\vec{u}, \vec{v}\})$.

(d) [1pt] Find a vector **NOT** in span($\{\mathbf{u}, \mathbf{v}\}$). Provide your reasons.

For example, $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ is not in span$(\{\vec{u}, \vec{v}\})$.

since $\begin{bmatrix} 5 & 1 \\ 1 & 1 \\ 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ has ~~not~~ no solution.

(e) [1pt] Find a vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

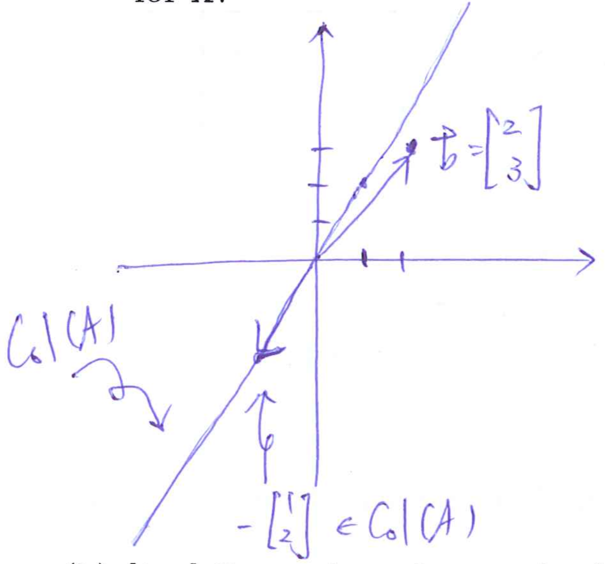
Solve $\begin{cases} 5x + y + 3z + w = 0 \\ x + y + z + w = 0 \end{cases} \Rightarrow \begin{cases} x + y + z + w = 0 \\ -4y - 2z - 4w = 0 \end{cases}.$

For example, $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is a solution.

2. Let

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

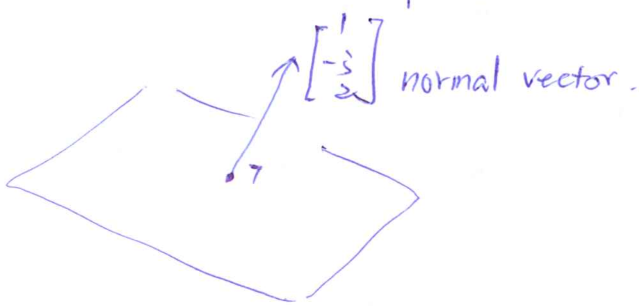
(a) [2pt] Draw the vector $\mathbf{b}$ and the subspace Col(A) in $\mathbb{R}^2$. Mark at least one vector in Col(A) in your drawing. Does $A\mathbf{x} = \mathbf{b}$ have a solution for $\mathbf{x}$?



No solution since $\vec{b} \notin$ Col(A).

(b) [1pt] Draw the subspace ker(A) and its normal vector in $\mathbb{R}^3$.

$$\ker(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - 3y + 2z = 0 \right\}$$



(c) [2pt] Find $\mathbf{h}_1$ and $\mathbf{h}_2$ such that ker$(A) = \text{span}(\{\mathbf{h}_1, \mathbf{h}_2\})$.

Solve. $x - 3y + 2z = 0$.

Set $y = 1, z = 0 \rightarrow \vec{h}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

Set $y = 0, z = 1 \Rightarrow \vec{h}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

3. Let

$$A = \begin{bmatrix} 1 & 5 & -4 & -2 \\ -1 & -5 & 4 & 3 \\ 2 & 10 & -8 & -6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -5 \\ 6 \\ -12 \end{bmatrix}.$$

(a) [2pt] Find the reduced echelon form of the augmented matrix $\left[\, A \,\middle|\, \mathbf{b} \,\right]$.

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} 1 & 5 & -4 & -2 & -5 \\ -1 & -5 & 4 & 3 & 6 \\ 2 & 10 & -8 & -6 & -12 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 5 & -4 & -2 & -5 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right]$$

$$\rightsquigarrow \underline{\left[\begin{array}{cccc|c} 1 & 5 & -4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]} \text{ is rref.}$$

(b) [3pt] Find $\mathbf{p}$, $\mathbf{h}_1$, and $\mathbf{h}_2$ such that the set of general solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\{\mathbf{p} + c_1\mathbf{h}_1 + c_2\mathbf{h}_2 : c_1, c_2 \in \mathbb{R}\}.$$

$$A\vec{x} = \vec{b} \Leftrightarrow \begin{cases} x_1 + 5x_2 - 4x_3 = -3 \\ x_4 = 1 \end{cases} \quad \cdots (*)$$

$$A\vec{x} = \vec{0} \Leftrightarrow \begin{cases} x_1 + 5x_2 - 4x_3 = 0 \\ x_4 = 0 \end{cases} \quad \cdots (**)$$

Free vas $= x_2, x_3$

Solve $(*)$:

Set $x_2 = x_3 = 0 \Rightarrow \underline{\vec{p} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}}.$

Solve $(**)$:

Set $x_2 = 1, x_3 = 0 \Rightarrow \underline{\vec{h}_1 = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$

Set $x_2 = 0, x_3 = 1 \Rightarrow \underline{\vec{h}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$

4. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of a *subspace* as clear as possible.

   (a) [3pt] Define what is a subspace and use a few sentences to explain the definition.

   (b) [2pt] Provide an example of a subspace and an example of a set that is not a subspace. Provide your reasons.

5. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ t \end{bmatrix}.$$

Find t such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	