國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

第二次期中考 November 14, 2022 **Midterm 2**

姓名 Name : ____________________

學號 Student ID # : Solution

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Find a 3×4 matrix A with $\text{rank}(A) = 1$ such that every entry of A is nonzero. Provide your reasons.

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Then $A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, So $\text{rank}(A) = 1$.

1 leading var.

2. [1pt] Find a 3×4 matrix A with $\text{null}(A) = 1$ such that every entry of A is nonzero. Provide your reasons.

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$. Then $A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so $\text{null}(A) = 1$.

1 free var

3. [1pt] Find a polynomial of degree at most 2 that is in $\text{span}\{1-x, 1-x^2\}$.

e.g. $1 - x^2$

or any $a + bx + cx^2$ with $a + b + c = 0$

4. [1pt] Find a polynomial of degree at most 2 that is NOT in $\text{span}\{1-x, 1-x^2\}$. e.g. x^2, since $p(1-x) + q(1-x^2) = x^2$ has no solution.

5. [1pt] Write $p(x) = 1 + x + x^2$ as a linear combination of $\{1, 1-x, (1-x)^2\}$.

Suppose
$$\begin{array}{rl} & a \cdot 1 \\ & b \cdot 1 - x \\ +) & c \cdot 1 - 2x + x^2 \\ \hline & 1 + x + x^2 \end{array}$$

Then $\begin{cases} a + b + c = 1 \\ -b - 2c = 1 \\ c = 1 \end{cases} \Rightarrow \begin{cases} a = 3 \\ b = -3 \\ c = 1 \end{cases}$

So $p(x) = 3 - 3(1-x) + 1 \cdot (1-x)^2$

6. Let

$$A = \begin{bmatrix} 1 & -3 & -2 & 1 \\ 2 & -6 & -4 & 3 \\ -5 & 15 & 10 & -10 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

such that R is the reduced echelon form of A.

(a) [1pt] Find a basis of Row(A).

$$\beta_R = \left\{ \begin{matrix} [1 \ \ -3 \ \ -2 \ \ 0], \\ [0 \ \ 0 \ \ 0 \ \ 1] \end{matrix} \right\}.$$

(b) [1pt] Find a basis of Col(A).

$$\beta_C = \left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix} \right\}$$

(c) [2pt] Find a basis of ker(A).

$$\begin{bmatrix} 1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free

Set $x_2 = 1$, $x_3 = 0 \Rightarrow \vec{h}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Set $x_2 = 0$, $x_3 = 1 \Rightarrow \vec{h}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

$\beta_K = \{ \vec{h}_1, \vec{h}_2 \}$.

(d) [1pt] Find a basis of ker($A^\top$).

Since $\dim(\ker(A^T)) = 3 - \text{rank}(A) = 3 - 2 = 1$, any nonzero vector in $\ker(A^T)$ form a basis.

Solve $A^T \vec{x} = \vec{0}$ and get one solution $\vec{x} = \begin{bmatrix} -5 \\ 5 \\ 1 \end{bmatrix}$.

$\Rightarrow \left\{ \begin{bmatrix} -5 \\ 5 \\ 1 \end{bmatrix} \right\}$ is a basis.

7. Let V be a subspace with a basis $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Let $\alpha = \{\mathbf{a}_1, \mathbf{a}_2\}$ be a subset in V. Suppose it is known that $\mathbf{a}_1 = \mathbf{b}_1 + 2\mathbf{b}_3$ and $\mathbf{a}_2 = \mathbf{b}_1 + 3\mathbf{b}_3$.

   (a) [1pt] Write $\mathbf{b}_1$ as a linear combination of $\{\mathbf{a}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

   Since $\vec{a}_1 = \vec{b}_1 + 2\vec{b}_3$, we know $\underline{\vec{b}_1 = \vec{a}_1 + 0\vec{b}_2 - 2\vec{b}_3}$.

   (b) [2pt] Show that $\{\mathbf{a}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is linearly independent. [No basis exchange lem]

   Suppose $c_1\vec{a}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = \vec{0}$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

   Then $c_1 \cdot (\vec{b}_1 + 2\vec{b}_3) + c_2\vec{b}_2 + c_3\vec{b}_3 = \vec{0}$

   $= c_1\vec{b}_1 + c_2\vec{b}_2 + (2c_1 + c_3)\vec{b}_3 = \vec{0}$

   Since β is a basis and is indep,

   we know $c_1 = c_2 = 2c_1 + c_3 = 0$. $\Rightarrow$ $c_1 = c_2 = c_3 = 0$.

   Therefore, $\{\vec{a}_1, \vec{b}_2, \vec{b}_3\}$ is indep.

   (c) [2pt] Find a basis S of V such that $\alpha \subseteq S \subseteq \alpha \cup \beta$.

   By applying basis exchange lemma $\cancel{\{\vec{a}_1}$ to β and the fact $\vec{a}_1 = \vec{b}_1 + 2\vec{b}_3$,

   $\{\vec{a}_1, \vec{b}_2, \vec{b}_3\} = \beta \cup \{\vec{a}_1\} \setminus \{\vec{b}_1\}$ is a basis of V.

   Apply the lemma to $\{\vec{a}_1, \vec{b}_2, \vec{b}_3\}$ with $\vec{a}_2 = \vec{b}_1 + 3\vec{b}_3$

   $= \vec{a}_1 + \vec{b}_3$,

   $\underline{\{\vec{a}_1, \cancel{b}\,\vec{a}_2, \vec{b}_2\}}$ is a basis of V.

8. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of a *basis* as clear as possible.

   (a) [3pt] Suppose V is a subspace. Define what is a basis of V and use a few sentences to explain the definition.

   (b) [2pt] Let $V = \mathbb{R}^3$. Provide an example of a basis of V and an example of a subset of V that is not a basis. Provide your reasons.

9. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find a basis of $\mathrm{Col}(A) \cap \mathrm{Col}(B)$.

$$\mathrm{Col}(A) = \left\{ \begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

$$\mathrm{Col}(B) = \left\{ \begin{bmatrix} a \\ b \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

So $\mathrm{Col}(A) \cap \mathrm{Col}(B) = \left\{ \begin{bmatrix} k \\ k \\ k \\ k \end{bmatrix} : k \in \mathbb{R} \right\}$

$$= \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

So $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathrm{Col}(A) \cap \mathrm{Col}(B)$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	