國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

第二次期中考 November 14, 2022 **Midterm 2**

姓名 Name : ____________________

學號 Student ID # : solution

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Find a 3×4 matrix A with $\text{rank}(A) = 2$ such that every entry of A is nonzero. Provide your reasons.

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$. Then $A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has $\text{rank}(A) = 2$.

2. [1pt] Find a 3×4 matrix A with $\text{null}(A) = 2$ such that every entry of A is nonzero. Provide your reasons.

Since A is 3×4, by dim thm we have $\text{rank}(A) + \text{null}(A) = 4$.

Thus, the A in the previous question is also an answer here.

3. [1pt] Find a polynomial of degree at most 2 that is in $\text{span}\{1 + x, 1 + x^2\}$.

e.g. $1 + x^2$.

or any $a + bx + cx^2$ with $a - b - c = 0$.

4. [1pt] Find a polynomial of degree at most 2 that is NOT in $\text{span}\{1 + x, 1 + x^2\}$.

e.g. x^2, since $p(1+x) + q(1+x^2) = x^2$ has no solution.

5. [1pt] Write $p(x) = 1 + x + x^2$ as a linear combination of $\{1, 1 + x, (1 + x)^2\}$.

Solve
$$\begin{array}{rl} & a \cdot 1 \\ & b \cdot 1 + x \\ +) & c \cdot 1 + 2x + x^2 \\ \hline & 1 + x + x^2. \end{array}$$

Then $\begin{cases} a + b + c = 1 \\ b + 2c = 1 \\ c = 1 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = -1 \\ c = 1 \end{cases}$.

So $p(x) = 1 \cdot 1 + (-1) \cdot (1 + x) + 1 \cdot (1 + x)^2$.

6. Let

$$A = \begin{bmatrix} 1 & -1 & 3 & -13 \\ -1 & 1 & -2 & 9 \\ 0 & 0 & -4 & 16 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

such that R is the reduced echelon form of A.

(a) [1pt] Find a basis of Row(A).

$$\beta_R = \left\{ \begin{matrix} [1 & -1 & 0 & -1] \\ [0 & 0 & 1 & -4] \end{matrix} \right\}.$$

(b) [1pt] Find a basis of Col(A).

$$\beta_C = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix} \right\}.$$

(c) [2pt] Find a basis of ker(A).

x_2, x_4 are free vars.

~~Set~~ Solve $R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$.

Set $x_2 = 1, x_4 = 0 \Rightarrow \vec{h}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Set $x_2 = 0, x_4 = 1 \Rightarrow \vec{h}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$.

$\beta_K = \{\vec{h}_1, \vec{h}_2\}$.

(d) [1pt] Find a basis of ker($A^\top$).

Since $\dim(\ker(A^T)) = 3 - \text{rank}(A) = 3 - 2 = 1$,

$\ker(A^T) = \text{span}\{\vec{v}\}$ for any nonzero vector ~~$\vec{v} \in \ker^T$~~ $\vec{v} \in \ker(A^T)$.

Solve $A^T \vec{x} = \vec{0} \Rightarrow$ one solution is $\begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$.

$\underline{\beta_L = \left\{ \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} \right\}}$

7. Let V be a subspace with a basis $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Let $\alpha = \{\mathbf{a}_1, \mathbf{a}_2\}$ be a subset in V. Suppose it is known that $\mathbf{a}_1 = \mathbf{b}_1 + 4\mathbf{b}_3$ and $\mathbf{a}_2 = \mathbf{b}_1 + 5\mathbf{b}_3$.

(a) [1pt] Write $\mathbf{b}_1$ as a linear combination of $\{\mathbf{a}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

Since $\vec{a}_1 = \vec{b}_1 + 4\vec{b}_3$, we know $\vec{b}_1 = \vec{a}_1 + 0\vec{b}_2 - 4\vec{b}_3$.

(b) [2pt] Show that $\{\mathbf{a}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is linearly independent.

Suppose $c_1\vec{a}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = \vec{0}$.

Then $c_1(\vec{b}_1 + 4\vec{b}_3) + c_2\vec{b}_2 + c_3\vec{b}_3 = \vec{0}$

$= c_1\vec{b}_1 + c_2\vec{b}_2 + (4c_1 + c_3)\vec{b}_3 = \vec{0}$.

Since β is indep, $c_1 = c_2 = 4c_1 + c_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$.

Thus, $\{\vec{a}_1, \vec{b}_2, \vec{b}_3\}$ is indep.

(c) [2pt] Find a basis S of V such that $\alpha \subseteq S \subseteq \alpha \cup \beta$.

Apply the basis exchange lemma to β with $\vec{a}_1 = \vec{b}_1 + 4\vec{b}_3$,

then $\beta \cup \{\vec{a}_1\} \setminus \{\vec{b}_1\} = \{\vec{a}_1, \vec{b}_2, \vec{b}_3\}$ is a basis of V.

Apply the lemma to $\{\vec{a}_1, \vec{b}_2, \vec{b}_3\}$ with $\vec{a}_2 = \vec{b}_1 + 5\vec{b}_3$

$= \vec{a}_1 + \vec{b}_3$;

then $\{\vec{a}_1, \vec{a}_2, \vec{b}_2\}$ is a basis of V.

8. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of a *basis* as clear as possible.

   (a) [3pt] Suppose V is a subspace. Define what is a basis of V and use a few sentences to explain the definition.

   (b) [2pt] Let $V = \mathbb{R}^3$. Provide an example of a basis of V and an example of a subset of V that is not a basis. Provide your reasons.

9. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find a basis of $\mathrm{Col}(A) \cap \mathrm{Col}(B)$.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	