國立中山大學　NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二）　**MATH 104 / GEAI 1209: Linear Algebra II**

期末考　June 6, 2022　**Final Exam**

姓名 Name : ____________________

學號 Student ID # : solution

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let A be a 3×4 matrix. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are orthogonal bases of $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, such that

$$[f_A]_\alpha^\beta = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $f_A : \mathbb{R}^4 \to \mathbb{R}^3$ is defined by $f_A(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^4$. Answer the following questions in terms of elements in α or in β.

(a) [1pt] Find $f(10\mathbf{v}_1 + 20\mathbf{v}_2 + 30\mathbf{v}_3)$.

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \\ 30 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 60 \\ 0 \end{bmatrix}, \text{ so } f(10\vec{v}_1+20\vec{v}_2+30\vec{v}_3) = \underline{40\vec{u}_1 + 60\vec{u}_2}$$

(b) [1pt] Find a vector $\mathbf{v}$ such that $f(\mathbf{v}) = 20\mathbf{u}_1 + 60\mathbf{u}_2$.

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 20 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \\ 0 \end{bmatrix}, \text{ so } \cancel{f}\ \cancel{\vec{v} = 20\vec{v}_1} +$$

$$\underline{\vec{v} = 5\vec{v}_1 + 20\vec{v}_2}$$

(c) [1pt] Find an element in $\ker(A)$.

e.g. $\underline{\vec{v}_3}$ (any $c_3\vec{v}_3 + c_4\vec{v}_4$ is okay).

(d) [1pt] Find an element in $\text{Col}(A)$.

e.g. $\underline{\vec{u}_1}$ (any $c_1\vec{u}_1 + c_2\vec{u}_2$ is okay.)

(e) [1pt] Find $\text{spec}(A^\top A)$.

~~Nonzero~~
singular values $= 4, 3$

$\Rightarrow \underbrace{\text{spec}(A^TA)}_{4\times 4} = \{4^2, 3^2, 0, 0\}$

$= \underline{\{16, 9, 0, 0\}}$

2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & -1 & -1 \\ 1 & 2 & -1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & -2 & -2 \end{bmatrix}.$$

(a) [1pt] Find an invertible matrix Q such that $Q^\top A Q = B$.

$$Q^\top = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ -1 & & 1 & \\ -1 & & & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & -1 & -1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

(b) [2pt] Find the inertia $(n_+(A), n_-(A), n_0(A))$ of A.

$$C = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & & \\ & -1 & -1 \\ & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$$

$$\mathrm{Iner}(C) = (0, 2, 1)$$

$$\mathrm{Iner}(A) = \mathrm{Iner}(B) = (1, 0, 0) + \mathrm{Iner}(C)$$

$$= (1, 2, 1)$$

3. [2pt] Let

$$M = \begin{bmatrix} 1 & 100 & 200 \\ 0 & 0 & 1 \\ 0 & -12 & 7 \end{bmatrix}.$$

Find an orthogonal matrix U such that $U^\top M U$ is an upper triangular matrix.

$\begin{bmatrix} 0 & 1 \\ -12 & 7 \end{bmatrix}$ has eigvals $3, 4$; eigvecs $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\Rightarrow$ Pick $W = \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} / \sqrt{10} \quad \begin{pmatrix} -3 \\ 1 \end{pmatrix} / \sqrt{10} \right]$

$$= \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{W} \\ 0 & & \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1/\sqrt{10} & -3/\sqrt{10} \\ & 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}.$$

4. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

(a) [3pt] Find an orthogonal matrix Q and a diagonal matrix D such that $Q^\top AQ = D$.

orthonormal

$\mathrm{null}(A) = 2 \Rightarrow \mathrm{spec}(A) = \{0, 0, x\}.$

since $0+0+x = \mathrm{tr}(A) = 1+1+4 = 6. \Rightarrow x = 6.$

"$\lambda = 0$" $\ker(A-\lambda I) = \ker\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} / \sqrt{2}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} / \sqrt{3} \right\}.$

"$\lambda = 6$" $\ker(A-\lambda I) = \ker\begin{bmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} / \sqrt{6} \right\}$

So $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$, $D = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 6 \end{bmatrix}$

(b) [2pt] Find q distinct values $\mu_1, \ldots, \mu_q$ and q projection matrices $P_1, \ldots, P_q$ such that

- $A = \sum_{j=1}^{q} \mu_j P_j$,
- $P_j^2 = P_j$ for any j,
- $P_i P_j = O$ for any $i \neq j$, and
- $\sum_{j=1}^{q} P_j = I$.

$\mu_1 = 0$, $P_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & -\frac{2}{6} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{2}{6} \\ -\frac{2}{6} & -\frac{2}{6} & \frac{2}{6} \end{bmatrix}$

$\mu_2 = 6$, $P_2 = \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

5. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是主成份分析（principal component analysis）。

請説明主成份的直觀意義、並描述主成份分析的功能；有必要的話可以加上一些圖來輔助説明。請解釋主成份分析的步驟，以及每一步其用意爲何。格式沒有限制，篇幅大約半面到一面。

(If Chinese is not your native language, you may use English or the language that you prefer.)

6. [extra 5pt] Let $x, y, x \in \mathbb{R}$ such that $x^2 + y^2 + z^2 = 1$. Find the maximum value of $2xy + 2yz$.

Hint: Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Observe that when $x^2+y^2+z^2 = 1$,

$$2xy + 2yz = R_A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right).$$

So the max is the largest eigval of A.

For A, ~~A charpoly~~ $s_0 = 1$

$s_1 = 0$

$s_2 = -2$

$s_3 = 0.$

$\Rightarrow$ charpoly $= 1(-x)^3 + 0(-x)^2 - 2(-x) + 0$

$= -x^3 + 2x = -x(x^2 - 2)$

$\Rightarrow$ $\mathrm{spec}(A) = \{-\sqrt{2}, 0, \sqrt{2}\}.$

<u>max $= \sqrt{2}$</u>

7. [extra 2pt] Let

$$A = \begin{bmatrix} 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Find spec(A).

$\text{null}(A-4I) \geq 2$

$\text{null}(A-3I) \geq 3$

$\Rightarrow \text{spec}(A) = \{4, 4, 3, 3, 3, \lambda_1, \lambda_2\}$.

Consider the equitable partition $\pi = (\{1,2,3\}, \{4,5,6,7\})$

$A/\pi = \begin{bmatrix} 4 & -4 \\ -3 & 3 \end{bmatrix}$.

$\Rightarrow \text{spec}(A/\pi) = \{0, 7\}$.

So $\underline{\text{spec}(A) = \{0, 3, 3, 3, 4, 4, 7\}}$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	