國立中山大學	NATIONAL SUN YAT-SEN UNIVERSITY
線性代數（二）	**MATH 104 / GEAI 1209: Linear Algebra II**
期末考 June 6, 2022	**Final Exam**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

– Enter your **Name** and **Student ID #** before you start.

– Using the calculator is not allowed (and not necessary) for this exam.

– Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.

– Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.

– 可用中文或英文作答

1. Let A be a 3×4 matrix. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are orthogonal bases of $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, such that

$$[f_A]_\alpha^\beta = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $f_A : \mathbb{R}^4 \to \mathbb{R}^3$ is defined by $f_A(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^4$. Answer the following questions in terms of elements in α or in β.

(a) [1pt] Find $f(10\mathbf{v}_1 + 20\mathbf{v}_2 + 30\mathbf{v}_3)$.

(b) [1pt] Find a vector $\mathbf{v}$ such that $f(\mathbf{v}) = 20\mathbf{u}_1 + 60\mathbf{u}_2$.

(c) [1pt] Find an element in $\ker(A)$.

(d) [1pt] Find an element in $\mathrm{Col}(A)$.

(e) [1pt] Find $\mathrm{spec}(A^\top A)$.

2. Let

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & 2 & 2 \\ -1 & 2 & 1 & 3 \\ -1 & 2 & 3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

(a) [1pt] Find an invertible matrix Q such that $Q^\top AQ = B$.

(b) [2pt] Find the inertia $(n_+(A), n_-(A), n_0(A))$ of A.

3. [2pt] Let

$$M = \begin{bmatrix} 1 & 100 & 200 \\ 0 & 0 & 1 \\ 0 & -10 & 7 \end{bmatrix}.$$

Find an orthogonal matrix U such that $U^\top MU$ is an upper triangular matrix.

4. Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

(a) [3pt] Find an orthogonal matrix Q and a diagonal matrix D such that $Q^\top AQ = D$.

(b) [2pt] Find q distinct values $\mu_1, \ldots, \mu_q$ and q projection matrices $P_1, \ldots, P_q$ such that

- $A = \sum_{j=1}^{q} \mu_j P_j$,
- $P_j^2 = P_j$ for any j,
- $P_i P_j = O$ for any $i \neq j$, and
- $\sum_{j=1}^{q} P_j = I$.

5. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是主成份分析（principal component analysis）。

   請說明主成份的直觀意義、並描述主成份分析的功能；有必要的話可以加上一些圖來輔助說明。請解釋主成份分析的步驟，以及每一步其用意爲何。格式沒有限制，篇幅大約半面到一面。

   (If Chinese is not your native language, you may use English or the language that you prefer.)

6. [extra 5pt] Let $x, y, z \in \mathbb{R}$ such that $x^2 + y^2 + z^2 = 1$. Find the maximum value of $2xy + 2yz$.

   Hint: Consider the matrix
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

7. [extra 2pt] Let

$$A = \begin{bmatrix} 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Find spec(A).

[END]

Page	Points	Score
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2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	