國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

期末考 June 6, 2022 **Final Exam**

姓名 Name : ____________________

學號 Student ID # : solution

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let A be a 3×4 matrix. Suppose $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are orthogonal bases of $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, such that

$$[f_A]_\alpha^\beta = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $f_A : \mathbb{R}^4 \to \mathbb{R}^3$ is defined by $f_A(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^4$. Answer the following questions in terms of elements in α or in β.

(a) [1pt] Find $f(10\mathbf{v}_1 + 20\mathbf{v}_2 + 30\mathbf{v}_3)$.

$[f_A]_\alpha^\beta \begin{bmatrix} 10 \\ 20 \\ 30 \\ 0 \end{bmatrix} = \begin{bmatrix} 50 \\ 40 \\ 0 \end{bmatrix} \Rightarrow f(10\vec{v}_1 + 20\vec{v}_2 + 30\vec{v}_3) = \underline{\underline{50\vec{u}_1 + 40\vec{u}_2}}$.

(b) [1pt] Find a vector $\mathbf{v}$ such that $f(\mathbf{v}) = 20\mathbf{u}_1 + 60\mathbf{u}_2$.

$[f_A]_\alpha^\beta \begin{bmatrix} 4 \\ 30 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \\ 0 \end{bmatrix} \Rightarrow \vec{v} = \underline{\underline{4\vec{v}_1 + 30\vec{v}_2}}$

(c) [1pt] Find an element in $\ker(A)$.

e.g. $\underline{\vec{v}_3}$ or any $c_3\vec{v}_3 + c_4\vec{v}_4$

(d) [1pt] Find an element in $\mathrm{Col}(A)$.

e.g. $\underline{\vec{u}_1}$ or any $c_1\vec{u}_1 + c_2\vec{u}_2$.

(e) [1pt] Find $\mathrm{spec}(A^\top A)$.

$\mathrm{spec}(A^TA) = \mathrm{spec}\left(\begin{bmatrix} 5 & & \\ & 2 & \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & & & 0 \\ & 2 & & 0 \\ & & 0 & 0 \end{bmatrix} \right)$

$= \underline{\underline{\{25, 4, 0, 0\}}}$

2. Let

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & 2 & 2 \\ -1 & 2 & 1 & 3 \\ -1 & 2 & 3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

(a) [1pt] Find an invertible matrix Q such that $Q^\top A Q = B$.

$$Q^T = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & & & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

(b) [2pt] Find the inertia $(n_+(A), n_-(A), n_0(A))$ of A.

$$\text{Let } C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow[\text{operation}]{\text{row/col}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \text{iner}(A) = \text{iner}(B) = \text{iner}(C) + (1,0,0) = (2,1,\cancel{0}1)$$

3. [2pt] Let

$$M = \begin{bmatrix} 1 & 100 & 200 \\ 0 & 0 & 1 \\ 0 & -10 & 7 \end{bmatrix}.$$

Find an orthogonal matrix U such that $U^\top M U$ is an upper triangular matrix.

$\begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix}$ has eigvals 2, 5, eigvecs $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, Pick $W = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} / \sqrt{5}$. orthogonal.

$$\text{Let } U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{W}} \\ 0 & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}.$$

4. Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

(a) [3pt] Find an orthogonal matrix Q and a diagonal matrix D such that $Q^\top A Q = D$.

$$\text{null}(A) = 2 \Rightarrow \text{spec}(A) = \{0, 0, x\}.$$

$$0 + 0 + x = \text{tr}(A) = 1 + 4 + 4 = 9 \Rightarrow x = 9.$$

"$\lambda = 0$" $\ker(A - \lambda I) = \ker \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \Big/ \sqrt{5}, \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix} \Big/ \sqrt{45} \right\}.$

"$\lambda = 9$" $\ker(A - \lambda I) = \ker \begin{bmatrix} -8 & 2 & 2 \\ 2 & -5 & 4 \\ 2 & 4 & -5 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Big/ 3 \right\}.$

$$\Rightarrow Q = \begin{bmatrix} 2/\sqrt{5} & 2/\sqrt{45} & 1/3 \\ -1/\sqrt{5} & 4/\sqrt{45} & 2/3 \\ 0 & -5/\sqrt{45} & 2/3 \end{bmatrix}$$

(b) [2pt] Find q distinct values $\mu_1, \ldots, \mu_q$ and q projection matrices $P_1, \ldots, P_q$ such that

- $A = \sum_{j=1}^{q} \mu_j P_j$,
- $P_j^2 = P_j$ for any j,
- $P_i P_j = O$ for any $i \neq j$, and
- $\sum_{j=1}^{q} P_j = I$.

$\mu_1 = 0$, $P_1 = \frac{1}{5}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{45}\begin{bmatrix} 4 & 8 & -10 \\ 8 & 16 & -20 \\ -10 & -20 & 25 \end{bmatrix}$

$$= \frac{1}{45}\begin{bmatrix} 40 & -10 & -10 \\ -10 & 25 & -20 \\ -10 & -20 & 25 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

$\mu_2 = 9$, $P_2 = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$

5. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是主成份分析（principal component analysis）。

   請說明主成份的直觀意義、並描述主成份分析的功能；有必要的話可以加上一些圖來輔助說明。請解釋主成份分析的步驟，以及每一步其用意爲何。格式沒有限制，篇幅大約半面到一面。

   (If Chinese is not your native language, you may use English or the language that you prefer.)

6. [extra 5pt] Let $x, y, x \in \mathbb{R}$ such that $x^2 + y^2 + z^2 = 1$. Find the maximum value of $2xy + 2yz$.

Hint: Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

See ver. A.

7. [extra 2pt] Let

$$A = \begin{bmatrix} 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Find spec(A).

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	