國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

第二次期中考 May 2, 2022 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis of $\mathbb{R}^3$. Let A be the 3×3 matrix such that

$$[f_A]_\beta^\beta = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Here $f_A : \mathbb{R}^3 \to \mathbb{R}^3$ is the function defined by $f_A(\mathbf{v}) = A\mathbf{v}$.

(a) [1pt] Write $A\mathbf{u}_1$ as a linear combination of β.

$$Au_1 = \underline{4u_1 + 0u_2 + 0u_3}$$

(b) [1pt] Write $A\mathbf{u}_3$ as a linear combination of β.

$$Au_3 = \underline{0u_1 + 1u_2 + 3u_3}$$

(c) [1pt] Find the characteristic polynomial of A.

$$\underline{P_A(x) = (-1)^3 (x-4)(x-3)^2}.$$

(d) [1pt] Find the minimal polynomial of A.

$$m_A(x) = \underline{(x-4)(x-3)^2}$$

(e) [1pt] Find the algebraic multiplicity and the geometric multiplicity of $\lambda = 3$.

$$\underline{a.m(\lambda) = 2}$$

$$\underline{g.m(\lambda) = \text{null} \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix} = 1}.$$

2. It is known that

$$\overset{D}{\begin{bmatrix}3 & 0\\0 & 4\end{bmatrix}} = \overset{Q^{-1}}{\begin{bmatrix}1 & 1\\3 & 4\end{bmatrix}^{-1}} \overset{A}{\begin{bmatrix}0 & 1\\-12 & 7\end{bmatrix}} \overset{Q}{\begin{bmatrix}1 & 1\\3 & 4\end{bmatrix}} \qquad Q^{-1} = \begin{bmatrix}4 & -1\\-3 & 1\end{bmatrix}$$

(a) [1pt] Find A^n.

$$A^n = QD^nQ^{-1}$$
$$= \begin{bmatrix}1 & 1\\3 & 4\end{bmatrix}\begin{bmatrix}3^n & \\ & 4^n\end{bmatrix}\begin{bmatrix}4 & -1\\-3 & 1\end{bmatrix}$$
$$= 3^n\cdot\begin{bmatrix}4 & -1\\12 & -3\end{bmatrix} + 4^n\cdot\begin{bmatrix}-3 & 1\\-12 & 4\end{bmatrix} = \underline{\begin{bmatrix}4\cdot 3^n - 3\cdot 4^n & -1\cdot 3^n + (1)\cdot 4^n\\ 12\cdot 3^n - 12\cdot 4^n & -3\cdot 3^n + 4\cdot 4^n\end{bmatrix}}$$

(b) [2pt] Solve the recurrence relation $a_{n+2} - 7a_{n+1} + 12a_n = 0$ with $a_0 = 1$ and $a_1 = 1$.

$$\begin{bmatrix}a_n\\a_{n+1}\end{bmatrix} = A\begin{bmatrix}a_{n-1}\\a_n\end{bmatrix} = \cdots = A^n\begin{bmatrix}a_0\\a_1\end{bmatrix}$$

$$a_n = \text{1st entry of } A^n\begin{bmatrix}1\\1\end{bmatrix} = \cancel{3\cdot 3^n + 3\cdot 4^n}$$
$$= \underline{3\cdot 3^n - 2\cdot 4^n}$$

(c) [2pt] Solve the differential equation $y'' - 7y + 12 = 0$, where y is a function of t.

Let $\vec{y} = \begin{bmatrix}y\\y'\end{bmatrix}$. Then $\dot{\vec{y}} = A\vec{y}$

$$\Rightarrow \dot{\vec{y}} = QDQ^{-1}\vec{y}$$
$$\Rightarrow (Q^{-1}\vec{y})^{\cdot} = D(Q^{-1}\vec{y})$$
$$\Rightarrow Q^{-1}\vec{y} = \begin{bmatrix}C_1e^{3t}\\C_2e^{4t}\end{bmatrix}$$
$$\Rightarrow y = \text{1st entry of } \vec{y} = Q\begin{bmatrix}C_1e^{3t}\\C_2e^{4t}\end{bmatrix}$$
$$= \underline{C_1e^{3t} + C_2e^{4t}}$$

3. Diagonalize the following matrix

$$A = \begin{bmatrix} -2 & 3 & -3 \\ -3 & 4 & -3 \\ -3 & 3 & -2 \end{bmatrix}.$$

That is, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$s_0 = 1$

$s_1 = -2+4-2 = 0$

$s_2 = \det\begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix} + \det\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} + \det\begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}$

$= 1 + 1 \cancel{+} -5 = -3$

$s_3 = \det A = 1 \cdot \det\begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix} = -2.$

$A \rightsquigarrow \begin{pmatrix} -2 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -5 & 3 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$\Rightarrow P_A(x) = (-x)^3 + 0(-x)^2 \cancel{+} -3(-x) - 2.$

$= -x^3 + 3x - 2. = -(x+2)(x-1)^2.$

If $P_A(x) = 0$, then $x = -2, 1, 1$.

Let $\lambda = -2$.

$A - (-2)I = \begin{bmatrix} 0 & 3 & -3 \\ -3 & 6 & -3 \\ -3 & 3 & 0 \end{bmatrix} \Rightarrow \ker(A+2I) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$

Let $\lambda = 1$.

$A - I = \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -3 & 3 & -3 \end{bmatrix} \Rightarrow \ker(A-I) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

4. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是一個矩陣的特徵值（eigenvalue）。

   請敘述特徵值的數學定義，並以自己的方式或是比喻來說明其直觀意義。請說明特徵值和特徵向量、特徵多項式之間的關係，以及如何計算特徵值。請給一些例子，並詳細解釋例子中的各個數學物件如何呼應你先前的說明。你也可以補充一些這個概念的相關性質（像是矩陣經過一些改變後，特徵值會如何改變、或是不會改變）；有必要的話可以加上一些圖來輔助說明。格式沒有限制，篇幅大約半面到一面。

   (If Chinese is not your native language, you may use English or the language that you prefer.)

5. [extra 2pt] Let A be the 9×9 matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let

$$\begin{aligned} p_A(x) &= \det(A - xI) \\ &= s_0(-x)^9 + s_1(-x)^8 + \cdots + s_9 \end{aligned}$$

be the characteristic polynomial of A. Find s_4.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	