國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

期末考 December 18, 2023 **Final Exam**

姓名 Name ： solution

學號 Student ID # ： ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **6 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto span $\left(\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}\right)$. Is f a bijection? **Provide your reasons.**

No. $\begin{bmatrix}1\\1\end{bmatrix}$ is not in the range, so f is not surjective.

2. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by 30°. Is f a bijection? **Provide your reasons.**

Yes. f^{-1} exist by rotation by 30° clockwisely.

3. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}x-y\\y-x\end{bmatrix}.$$

Find range(f).

$$\text{range}(f) = \left\{ \begin{bmatrix}x-y\\y-x\end{bmatrix} : x, y \in \mathbb{R} \right\} = \text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}.$$

4. [1pt] Let $\mathcal{P}_d$ be the vector space of polynomials of degree at most d. Let $f : \mathcal{P}_2 \to \mathcal{P}_3$ be the function defined by $p \mapsto x \cdot p + p'$, where p' is the derivative of the polynomial p. Is f a linear function? **Provide your reasons.**

Yes.
$$f(p_1+p_2) = x\cdot(p_1+p_2) + (p_1+p_2)'$$
$$= xp_1 + xp_2 + p_1' + p_2' = f(p_1) + f(p_2)$$
$$f(r\cdot p) = x\cdot(rp) + (rp)' = r\cdot xp + rp' = r\cdot f(p).$$

5. [1pt] Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $x \mapsto x^2$. Is f a linear function? **Provide your reasons.**

No. $f(2x) = 4x^2 \neq 2\cdot f(x)$.

6. Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \ \mathbf{b}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

Let $\alpha = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases of

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}.$$

(a) [1pt] Find the vector $\mathbf{u}$ such that $[\mathbf{u}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$[\vec{u}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ means $\vec{b}_1 + \vec{b}_2 = \underline{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$

(b) [1pt] Find $[\mathbf{v}]_\alpha$.

$\vec{v} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, so $[\vec{v}]_\alpha = \underline{\begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}}$.

(c) [1pt] Find $[\mathbf{v}]_\beta$.

$\vec{v} = 2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 2\vec{b}_1 - 3\vec{b}_2$, so $[\vec{v}]_\beta = \underline{\begin{bmatrix} 2 \\ -3 \end{bmatrix}}$.

(d) [2pt] Find the change of basis matrix $[\mathrm{id}]_\alpha^\beta$ from α to β.

$[id]_\alpha^\beta = \begin{bmatrix} | & | \\ [\vec{a}_1]_\beta & [\vec{a}_2]_\beta \\ | & | \end{bmatrix} = \underline{\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}}$.

7. Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ be a function defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - 5x_2 - 3x_3 - 2x_4 + 9x_5 \\ x_4 - 4x_5 \\ 5x_1 - 25x_2 - 15x_3 - 13x_4 + 57x_5 \end{bmatrix}$$

(a) [2pt] Find a matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^5$.

(b) [3pt] Find a basis of $\ker(f)$.

See ver. A.

8. [5pt] Mathematical essay: Write a few paragraphs to introduce *isomorphism.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

9. [extra 5pt] Let

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be bases of $\mathbb{R}^3$. Given that

$$[f]_\beta^\beta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

find $[f]_\alpha^\alpha$.

See ver. A.

10. [extra 2pt] Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

be the vector space of all 2×2 matrices. Let

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\beta = \{E_1, E_2, E_3, E_4\}$ is a basis of V. Define a function $f : V \to V$ by

$$X \mapsto \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} X.$$

Find the matrix representation $[f]_\beta^\beta$ of the function f with respect to the bases β and β.

See ver A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	