國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

第二次期中考 November 13, 2023 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 = 0 \right\}$. Is V a subspace in $\mathbb{R}^2$? Provide your reason.

Yes, The set $V = \{(0,0)\}$ satisfies the three conditions

closed under "+", closed under "·", has zero.

2. [1pt] Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = 1 \right\}$. Is V a subspace in $\mathbb{R}^2$? Provide your reason.

No. We have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in V$ but

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \notin V$.

3. Let

$$A = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 4 & 4 & 6 & 6 \\ 5 & 5 & 1 & 1 \end{bmatrix}.$$

(a) [1pt] Find a basis of Row(A).

$\{(2,2,3,3), (5,5,1,1)\}$

or $\{(1,1,0,0), (0,0,1,1)\}$

(b) [1pt] Find a basis of Col(A).

$\left\{ \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}$.

(c) [1pt] Find rank(A) and null(A).

rank$(A) = 2$

null$(A) = 2$

4. [3pt] Let

$$S = \{(x-1)(x-3), (x-1)(x-5), (x-3)(x-5)\}.$$

Show that S is linearly independent, or provide a certificate of S not being independent.

It is linearly independent.

Suppose $c_1(x-1)(x-3) + c_2(x-1)(x-5) + c_3(x-3)(x-5) = 0$.

Plug in $x=1$. Then $c_3 = 0$.

$x=3$ Then $c_2 = 0$.

$x=5$ Then $c_1 = 0$.

So S is linearly independent.

5. [2pt] Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ be vectors in $\mathbb{R}^3$. Suppose we know

$$\operatorname{span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}) = \operatorname{span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}).$$

What is the relation between $\mathbf{u}_4$ and the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$? Provide an example to demonstrate this behavior.

We know $u_4 \in \operatorname{span}(\{u_1, u_2, u_3\})$

For example, $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $u_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then $\operatorname{span}(\{u_1, u_2, u_3\})$
$= \operatorname{span}(\{u_1, u_2, u_3, u_4\}) = \mathbb{R}^3$.

6. [5pt] Let

$$A = \begin{bmatrix} 1 & -1 & -5 & 3 & 7 \\ 2 & -2 & -10 & 7 & 15 \\ -4 & 4 & 20 & -13 & -29 \end{bmatrix}.$$

Find a basis of $\ker(A)$.

$$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & -5 & 3 & 7 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & -1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

leading free

Set $x_2 = 1$, $x_3 = 0$, $x_5 = 0$. Then $\vec{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Set $x_2 = 0$, $x_3 = 1$, $x_5 = 0$. Then $\vec{n}_2 = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Set $x_2 = 0$, $x_3 = 0$, $x_5 = 1$. Then $\vec{n}_3 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

Then $S = \{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is a basis of $\ker(A)$.

7. [5pt] Mathematical essay: Write a few paragraphs to introduce the notion of a basis.

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

8. [extra 2pt] Let V be the space of all functions defined on $(0, \infty)$. Determine if $S = \{\ln(x), \ln(x^2)\}$ is linearly independent or not.

No.

$\ln(x^2) = 2 \cdot \ln(x)$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	