國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

**線性代數（二）** **MATH 104A / GEAI 1209A: Linear Algebra II**

**期末考** May 29, 2023 **Final Exam**

姓名 Name : ____________________

學號 Student ID # : ____________________

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| Lecturer: | Jephian Lin 林晉宏 |
| Contents: | cover page, |
| | **6 pages** of questions, |
| | score page at the end |
| To be answered: | on the test paper |
| Duration: | **110 minutes** |
| Total points: | **20 points** + 7 extra points |

**Do not open this packet until instructed to do so.**

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Find a real matrix that is diagonalizable but some of its eigenvalue is not real. Provide your reasons.

2. [1pt] Find a real matrix that is diagonalizable but its eigenbasis is not mutually orthogonal. Provide your reasons.

3. [1pt] Find a real matrix that is not diagonalizable. Provide your reasons.

4. [2pt] Find the spectral decomposition of

$$\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

5. [5pt] Let

$$A = \begin{bmatrix} -17 & 3 & 2 & 1 \\ -73 & 12 & 9 & 5 \\ -57 & 10 & 6 & 4 \\ -41 & 7 & 5 & 2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}.$$

It is known that $\mathbf{v}$ is an eigenvector of $A$. Find the spectrum of $A$.

6. [5pt] Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 3 \end{bmatrix}.$$

Find the spectrum of $A$. You may also give some partial information of the spectrum of $A$ to get some partial credits, e.g., $\lambda_1 \geq 10$.

7. Let $A$ be an $m \times n$ matrix and $A = U\Sigma V^\top$ its singular value decomposition. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of the *singular value decomposition* as clear as possible.

   (a) [2pt] Explain what $U$, $\Sigma$ and $V$ are, including their sizes and properties.

   (b) [2pt] Let $\mathbf{u}_i$'s and $\mathbf{v}_i$'s be the columns of $U$ and $V$, respectively. Describe their relations.

   (c) [1pt] Give some reasons about why the singular value decomposition is important.

8. [extra 5pt] Let $A$ be the $9 \times 9$ matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Find the inertia of $A$.

9. [extra 2pt] Let

$$f(x, y) = x^2 + 4xy + 4y^2.$$

Find the maximum value of $f(x, y)$ subject to $x^2 + y^2 = 1$.

**[END]**

| Page | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 5 | |
| 6 | 2 | |
| Total | 20 (+7) | |