國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

期末考 May 29, 2023 **Final Exam**

姓名 Name : ____________________

學號 Student ID # : Solution

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **6 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Find a real matrix that is diagonalizable but some of its eigenvalue is not real. Provide your reasons.

$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ ~~char~~ $P_A(x) = x^2+2$

$\mathrm{spec}(A) = \{\sqrt{2}i, -\sqrt{2}i\}$

2. [1pt] Find a real matrix that is diagonalizable but its eigenbasis is not mutually orthogonal. Provide your reasons.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ $\mathrm{spec}(A) = \{0, 2\}$.

eigvecs $= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

3. [1pt] Find a real matrix that is not diagonalizable. Provide your reasons.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\mathrm{spec}(A) = \{1, 1\}$.

$\begin{cases} a.m(1) = 2 \\ g.m(1) = 1 \end{cases}$

4. [2pt] Find the spectral decomposition of

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

$P_A(x) = x^2 - 2x - 24 = (x-6)(x+4)$

$\lambda_1 = 6 \to u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2}$

$\lambda_2 = -4 \to u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{2}$

$\to A = 6 \cdot \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + (-4) \cdot \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

5. [5pt] Let

$$A = \begin{bmatrix} -17 & 3 & 2 & 1 \\ -73 & 12 & 9 & 5 \\ -57 & 10 & 6 & 4 \\ -41 & 7 & 5 & 2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}$$

$-17+12+6+2=3$
$-73+48+27+10=12$
$-57+40+18+8=9$
$-41+28+15+4=6$

It is known that $\mathbf{v}$ is an eigenvector of A. Find the spectrum of A.

Let $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$. Then $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$

Thus, $Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 2 & 1 \\ 12 & 12 & 9 & 5 \\ 9 & 10 & 6 & 4 \\ 6 & 7 & 5 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Since $\operatorname{spec}\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) = \{-1, -1, 2\}$.

$\operatorname{Spec}(A) = \{3, -1, -1, 2\}$.

6. [5pt] Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 3 \end{bmatrix}.$$

Find the spectrum of A. You may also give some partial information of the spectrum of A to get some partial credits, e.g., $\lambda_1 \geq 10$.

① $\lambda = -1$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 0 & 0 \\ 1 & 1 & 0 & 4 & 0 \\ 1 & 1 & 0 & 0 & 4 \end{bmatrix} \quad \text{null} \geq 1$$

② $\lambda = 3$

$$A - 3I = \begin{bmatrix} -3 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{null} \geq 2.$$

$\to$ spec $(A) = \{-1, 3, 3, ?, ?\}$

③ equitable partition

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 3 & & \\ 1 & 1 & & 3 & \\ 1 & 1 & & & 3 \end{array}\right] \to$$

qutient mtx

$$B = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$$

$P_B(x) = x^2 - 4x - 3$

roots $= \dfrac{4 \pm \sqrt{16+12}}{2} = 2 \pm \sqrt{7}$

So $\text{spec}(A) = \{-1, 3, 3, 2 \pm \sqrt{7}\}$.

7. Let A be an $m \times n$ matrix and $A = U\Sigma V^\top$ its singular value decomposition. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of the *singular value decomposition* as clear as possible.

   (a) [2pt] Describe the shape and the properties of matrices U, Σ and V. For example, M is an $m \times m$ diagonal matrix.

   (b) [2pt] Let $\mathbf{u}_i$'s and $\mathbf{v}_i$'s be the columns of U and V, respectively. Describe their relations.

   (c) [1pt] Give some reasons about why the singular value decomposition is important.

8. [extra 5pt] Let A be the 9×9 matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Find the inertia of A.

See ver. A.

9. [extra 2pt] Let

$$f(x,y) = x^2 + 4xy + 4y^2.$$

Find the maximum value of $f(x,y)$ subject to $x^2 + y^2 = 1$.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	