國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第一次期中考 March 20, 2023 **Midterm 1**

姓名 Name : Solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Write down the 3×3 elementary matrix for the row operation $\rho_1 : \times 2$ and find its determinant.

$$E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det(E) = 2.$$

2. [1pt] Write down the 3×3 elementary matrix for the row operation $\rho_3 : +4\rho_2$ and find its determinant.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad \det(E) = 1.$$

3. [1pt] Write down the 3×3 elementary matrix for the row operation $\rho_1 \leftrightarrow \rho_3$ and find its determinant.

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \det(E) = -1$$

4. [2pt] Find the adjugate of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The i,j-entry of A^{adj} = ~~ij~~ j,i-cofactor

$= (-1)^{i+j} \cdot \det(A(j,i))$.

So

$$A^{adj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

5. [2pt] Find the determinant of

add row 2~6 to row 1.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

$\rho_i := -\rho_1$ for $i = 2 \sim 6$.

$$\det(A) = \det\begin{bmatrix} 5 & \cdots & 5 \\ 0 & \ddots & 1 \\ 1 & \ddots & 0 \end{bmatrix} \underset{\rho_1 : \times \frac{1}{5}}{=} 5 \cdot \det\begin{bmatrix} 1 & \cdots & 1 \\ 0 & \ddots & 1 \\ 1 & & 0 \end{bmatrix} = -5 \det\begin{bmatrix} 1 & \cdots & 1 \\ & -1 & 0 \\ 0 & & -1 \end{bmatrix}$$

$$= -5$$

6. [3pt] Find the determinant of

$$L = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Let $L_n = \begin{bmatrix} 2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2 \end{bmatrix}$. Then $L = L_7$.

By Laplace expansion along the 1st row,

$$\det(L_n) = 2 \cdot \det(L_{n-1}) - \det\begin{bmatrix} 1 & 1 & & 0 \\ 0 & 2 & \ddots & \\ \vdots & 1 & \ddots & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$= 2 \cdot \det(L_{n-1}) - \det(L_{n-2}).$$

$\det(L_1) = \det[2] = 2$.

$\det(L_2) = \det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3$.

$\det(L_3) = 2 \cdot 3 - 2 = 4$ $\cdots$ $\det(L) = \det(L_7) = 8$.

7. Let

$$A_x = \begin{bmatrix} -x & 1 & 1 & 1 & 1 \\ 1 & -x & 0 & 0 & 0 \\ 1 & 0 & -x & 0 & 0 \\ 1 & 0 & 0 & -x & 0 \\ 1 & 0 & 0 & 0 & -x \end{bmatrix}.$$

(a) [2pt] Find $\det(A_x)$.

See ver. A.

(b) [3pt] Find all x such that $\det(A_x) = 0$. For each of such x, find a nonzero vector $\mathbf{v}$ in $\ker(A_x)$.

See ver A.

8. [5pt] Let

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}.$$

Let E be a 2×2 elementary matrix. Discuss how E change the shape of S into $ES = \{E\mathbf{v} : \mathbf{v} \in S\}$ and calculate its area. Make sure you consider each of the three types of elementary matrices and give some concrete examples.

9. [extra 2pt] Let

$$A = \begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{y} & - \\ - & \mathbf{z} & - \end{bmatrix} \text{ and } B = \begin{bmatrix} - & \mathbf{x}+\mathbf{y} & - \\ - & \mathbf{y}+2\mathbf{z} & - \\ - & \mathbf{z}+3\mathbf{x} & - \end{bmatrix}$$

be 3×3 matrices. Suppose $\det(A) = 1$. Find $\det(B)$.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	