國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第二次期中考 April 24, 2023 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

(a) [1pt] Find an eigenvector of A and write down its corresponding eigenvalue.

$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\lambda = 2$.

(b) [1pt] Find a nonzero vector that is not an eigenvector of A.

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) [1pt] Find A^{100}.

$$A^{100} = \begin{bmatrix} 2^{100} & & \\ & 0 & \\ & & (-3)^{100} \end{bmatrix}$$

(d) [1pt] Find the characteristic polynomial of A.

$$\det(A - xI) = (2-x)(-x)(-3-x)$$
$$= -x(x-2)(x-3)$$
$$= -x^3 + 5x^2 - 6x$$

(e) [1pt] Find the spectrum of A.

$\mathrm{spec}(A) = \{2, 0, -3\}$

2. Suppose we know that

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/5 & 1/5 \\ 4/5 & -1/5 \end{bmatrix}}_{Q^{-1}}.$$

(a) [3pt] Find the formula of a_n in the recurrence relation

$$\begin{cases} a_{n+2} &= 3a_{n+1} + 4a_n, \\ a_0 &= 1, \ a_1 = 0. \end{cases}$$

Let $\vec{x}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. Then $\vec{x}_{n+1} = A \cdot \vec{x}_n$

Thus, $\vec{x}_n = A^n \cdot \vec{x}_0 = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$A^n = Q D^n Q^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 4^n & \\ & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$

a_n = 1st entry of $\vec{x}_n$ = 1st entry of $A^n \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = 1,1-entry of A^n

$= \frac{1}{5} \cdot 4^n + \frac{4}{5}(-1)^n$.

(b) [2pt] Let x_1 and x_2 be functions in t. Solve the system of differential equations. (Remember to add constants at appropriate places.)

$$\begin{cases} \dot{x}_1 &= \quad x_2, \\ \dot{x}_2 &= 4x_1 + 3x_2. \end{cases}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$. Then $\dot{\vec{x}} = A\vec{x}$.

~~Let $Q\vec{y} = \vec{x}$.~~ Then $\dot{\vec{x}} = Q \cancel{A} D Q^{-1} \vec{x}$

$(Q^{-1}\dot{\vec{x}}) = Q^{-1}\dot{\vec{x}} = D(Q^{-1}\vec{x})$

Let $Q^{-1}\vec{x} = \vec{y}$ $\Updownarrow$ $\vec{x} = Q\vec{y}$

$\dot{\vec{y}} = D\vec{y}$

$\begin{cases} \dot{y}_1 = 4y_1 \\ \dot{y}_2 = -y_2 \end{cases} \Rightarrow \begin{cases} y_1 = C_1 e^{4t} \\ y_2 = C_2 e^{-t}. \end{cases}$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 e^{4t} + C_2 e^{-t} \\ C_1 4 e^{4t} - C_2 e^{-t} \end{bmatrix}$

3. [5pt] Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Find the characteristic polynomial and the spectrum of A.

$s_0 = 1$

$s_1 = \cancel{0}\ tr(A) = 0$.

$s_2 = 4 \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -4$.

$s_3 = 0$ since every principal 3×3 submtx has repeated rows.

$s_4 = \cancel{0 \text{ sin}}$ $\det(A) = 0$ since A has repeated rows.

$\Rightarrow \; P_A(x) = (-x)^4 + 0(-x)^3 + (-4)(-x)^2 + 0(-x) + 0$.

$= x^4 - 4x^2$.

since $P_A(x) = x^2(x^2-4) = x^2(x+2)(x-2)$.

$spec(A) = \{0, 0, 2, -2\}$.

4. [5pt] Suppose we know that

$$\overset{A}{\begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}} = \overset{Q\ \cancel{D}}{\begin{bmatrix} 3/5 & \cancel{-}4/5 \\ -4/5 & 3/5 \end{bmatrix}} \overset{D}{\begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix}} \overset{Q^{-1} = Q^T}{\begin{bmatrix} 3/5 & -4/5 \\ \cancel{-}4/5 & 3/5 \end{bmatrix}}.$$

Describe the ellipse defined by

$$73x^2 + 72xy + 52y^2 = 1,$$

including the directions and the length of its axes. Provide reasons to your answers.

Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$. Then the equation is the same as.

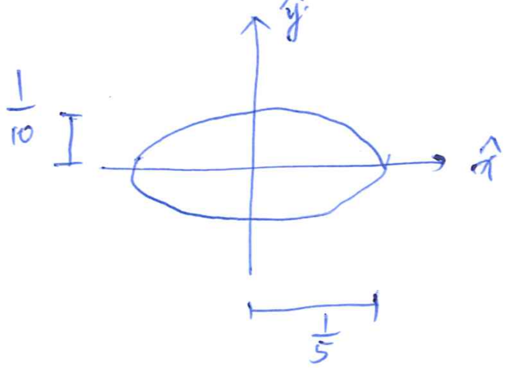
$$u^T A u = 1.$$

$$\Updownarrow$$

$$u^T Q D Q^{-1} u$$

Let $\hat{u} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$ be $= Q^T u$. Equivalently, $u = Q\hat{u}$.

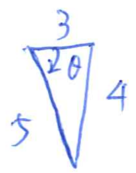
Then $\hat{u}^T D \hat{u} = 1 \Leftrightarrow 25\hat{x}^2 + 100\hat{y}^2 = 1$.

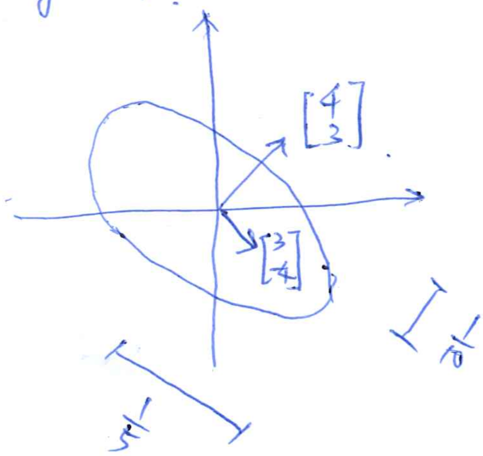


$\begin{bmatrix} x \\ y \end{bmatrix} = Q \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$

$\Longrightarrow$

Q = rotation mtx of θ.





5. [extra 2pt] Let A be the 10×10 matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let

$$\det(A - xI) = a_0 x^{10} + a_1 x^9 + a_2 x^8 + \cdots + a_{10}$$

be its characteristic polynomial. Find a_2.

$a_2 = S_2 =$ sum of 2×2 p-minors.

$= \binom{10}{2} \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$= \underline{-45}.$

6. [extra 5pt] Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

Observe that $A+I$ has nullity 2.

So ~~0, 0,~~ -1 is an eigval of A of multiplicity 2.

Since $\operatorname{tr}(A) = 0$, so ~~the eigen~~ $\operatorname{spec}(A) = \{-1, -1, 2\}$.

$\lambda = 2 \Rightarrow \ker \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$

$\lambda = -1 \Rightarrow \ker \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$

Choose $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & & \\ & -1 & \\ & & -1 \end{bmatrix}$

(b) Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^{\top}$.

By normalizing each columns of Q,

we may choose

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}, \quad D = \begin{bmatrix} 2 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
6	5	
Total	20 (+7)	