國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第二次期中考 April 24, 2023 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **6 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) [1pt] Find an eigenvector of A and write down its corresponding eigenvalue.

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda = 4.$$

(b) [1pt] Find a nonzero vector that is not an eigenvector of A.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(c) [1pt] Find A^{100}.

$$A^{100} = \begin{bmatrix} 4^{100} & & \\ & (-1)^{100} & \\ & & 0 \end{bmatrix}$$

(d) [1pt] Find the characteristic polynomial of A.

$$\begin{aligned} \det(A - xI) &= (4-x)(-1-x)(-x) \\ &= -x(x-4)(x+1) \\ &= -x^3 + 3x^2 + 4x \end{aligned}$$

(e) [1pt] Find the spectrum of A.

$$\operatorname{spec}(A) = \{4, -1, 0\}$$

2. Suppose we know that

$$\begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{bmatrix}.$$

(a) [3pt] Find the formula of a_n in the recurrence relation

$$\begin{cases} a_{n+2} &= -a_{n+1} + 6a_n, \\ a_0 &= 1, \ a_1 = 0. \end{cases}$$

Let $\vec{x}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. Then $\vec{x}_{n+1} = A \cdot \vec{x}_n$

Thus, $\vec{x}_n = A^n \cdot \vec{x}_0 = A^n \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$A^n = QD^nQ^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 2^n & \\ & (-3)^n \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

$a_n =$ 1st entry of $\vec{x}_n =$ 1st entry of $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1,1)$-entry of A^n

$= \cancel{\frac{1}{5}} \cdot \frac{3}{5} \cdot 2^n + \frac{2}{5} \cdot (-3)^n$.

(b) [2pt] Let x_1 and x_2 be functions in t. Solve the system of differential equations. (Remember to add constants at appropriate places.)

$$\begin{cases} \dot{x}_1 &= \quad\quad x_2, \\ \dot{x}_2 &= 6x_1 \ -x_2. \end{cases}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $\dot{\vec{x}} = A\vec{x}$

$\dot{\vec{x}} = Q D Q^{-1} \vec{x}$

$(\dot{Q^{-1}\vec{x}}) = Q^{-1}\dot{\vec{x}} = D(Q^{-1}\vec{x})$

Let $\vec{y} = Q^{-1}\vec{x}$, $\vec{x} = Q\vec{y}$ $\Rightarrow$ $\dot{\vec{y}} = D\vec{y}$

$\begin{cases} \dot{y}_1 = 2y_1 \\ \dot{y}_2 = -3y_2 \end{cases} \Rightarrow \begin{cases} y_1 = C_1 e^{2t} \\ y_2 = C_2 e^{-3t}. \end{cases}$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 e^{2t} + C_2 e^{-3t} \\ C_1 2 e^{2t} - C_2 3 e^{-3t} \end{bmatrix}$

3. [5pt] Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Find the characteristic polynomial and the spectrum of A.

See ver. A.

4. [5pt] Suppose we know that

$$\begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix} = \begin{bmatrix} 3/5 & +4/5 \\ -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ +4/5 & 3/5 \end{bmatrix}.$$

Describe the ellipse defined by

$$73x^2 + 72xy + 52y^2 = 1,$$

including the directions and the length of its axes. Provide reasons to your answers.

See ver. A.

5. [extra 2pt] Let A be the 10×10 matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let

$$\det(A - xI) = a_0x^{10} + a_1x^9 + a_2x^8 + \cdots + a_{10}$$

be its characteristic polynomial. Find a_2.

See ver A.

6. [extra 5pt] Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

See ver. A.

(b) Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^{\top}$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
6	5	
Total	20 (+7)	