國立中山大學

NATIONAL SUN YAT-SEN UNIVERSITY

線性代數 (一)

MATH 103A / GEAI 1215A: Linear Algebra I

期末考

December 25, 2024

Final Exam

姓名 Name: Solution

學號 Student ID # : _____

Lecturer: Jephian Lin 林晉宏

Contents: cover page,

6 pages of questions,

score page at the end

To be answered: on the test paper

Duration: 110 minutes

Total points: 20 points + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

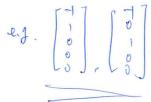
- Enter your Name and Student ID # before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining** it or circling it. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let

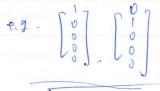
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 4 & 4 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is known that R is an echelon form of A.

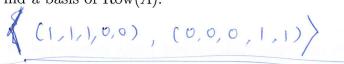
(a) [1pt] Find at least two elements in ker(A).



(b) [1pt] Find at least two elements **not** in ker(A).



(c) [1pt] Find a basis of Row(A).



(d) [1pt] Find a basis of ker(A).

(e) [1pt] Find a basis of Col(A).

$$\left\langle \begin{bmatrix} \frac{1}{2} \\ \frac{3}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{4}{3} \end{bmatrix} \right\rangle$$

2. Let $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the vector space of polynomial of degree at most 2. Let

$$B_1 = \langle 1, x, x^2 \rangle, B_2 = \langle 1, (x+1), (x+1)^2 \rangle, B_3 = \langle f_1(x), f_2(x), f_3(x) \rangle,$$

where

$$f_1(x) = \frac{(x-3)(x-5)}{(1-3)(1-5)}, f_2(x) = \frac{(x-1)(x-5)}{(3-1)(3-5)}, f_3(x) = \frac{(x-1)(x-3)}{(5-1)(5-3)}.$$

It is known that each of B_1 , B_2 , B_3 is a basis of \mathcal{P}_{3} . Let

$$p(x) = 1 + 2x + 3x^2 \text{ and } \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) [1pt] Find $\operatorname{Rep}_{B_1}(p(x))$. $p(x) = 1 \cdot 1 + 2 \cdot 7 + 3 \cdot x^2$, so $\operatorname{Rep}(p(x)) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) [1pt] Find
$$\operatorname{Rep}_{B_2}(p(x))$$
.
 $p(x) = 2 - 4(x+1) + 3 \cdot (x+1)^2$, so $\operatorname{Rep}_{B_2}(p(x)) = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$

(c) [1pt] Find
$$\operatorname{Rep}_{B_3}(p(x))$$
.
Note that $p(x) = p(1) \cdot f_1(x) + p(3) \cdot f_2(x) + p(5) \cdot f_3(x)$

$$= 6 f_1 + 34 f_2 + 81 f_3$$

$$\leq 6 \operatorname{Rep}_{B_3}(p(x)) = \begin{bmatrix} 6 \\ 34 \\ 86 \end{bmatrix}$$

(d) [1pt] Find a polynomial q(x) such that $Rep_{B_2}(q(x)) = \mathbf{u}$.

(e) [1pt] Find a polynomial q(x) such that $Rep_{B_3}(q(x)) = \mathbf{u}$.

3. Let $\mathcal{M}_{2\times 2}$ be the vector space of all 2×2 real matrices. Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

be a subspace of $\mathcal{M}_{2\times 2}$.

(a) [1pt] Find a basis of $\mathcal{M}_{2\times 2}$.

(b) [1pt] Find at least two elements in V.

(c) [3pt] Find a basis of V. Hint: Write down the equations for a, b, c, d.

4. [5pt] Mathematical essay: Write a few paragraphs to introduce dimension.

Your score will be based on the following criteria.

- The definition is clear.
- Some sentences are added to explain the definition.
- Examples or pictures are included to help understanding.
- The sentences are complete.

5. Let $B = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \rangle$ be a basis of a vector space V. Suppose $\mathbf{p} = \mathbf{u}_1 + \mathbf{u}_4$. Let

$$B_1 = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{p}, \mathbf{u}_4 \rangle$$
, and $B_2 = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{p} \rangle$.

(a) [extra 1pt] Explain why B_1 is not a basis of V.

(b) [extra 2pt] Show that B_2 is independent.

Suppose
$$G_{1}U_{1}+G_{2}U_{2}+G_{3}U_{3}+G_{4}p=0$$

Then $G_{1}U_{1}+G_{2}U_{2}+G_{3}U_{3}+G_{4}U_{4}=0$

and $G_{1}+G_{2}U_{1}+G_{2}U_{2}+G_{3}U_{3}+G_{4}U_{4}=0$.

Since B is a basis, $G_{1}+G_{2}=0$ $G_{2}=0$ $G_{3}=0$ $G_{4}=0$.

(c) [extra 2pt] Show that $span(B_2) = V$.

Let
$$\vec{V} = C_1\vec{U}_1 + C_2\vec{U}_2 + C_3\vec{U}_3 + C_4\vec{U}_4$$
 be an element $\vec{v}_1 \vec{V}_2$.

Then $\vec{V} = (G - C_4)\vec{U}_1 + C_2\vec{U}_2 + C_3\vec{V}_3 + C_4\vec{p} \in Span(B_2)$.

So $Span(B_2) = \vec{V}_2$.

6. [extra 2pt] Let

$$V = \{(x+1)(a+bx+cx^2) : a, b, c \in \mathbb{R}\}.$$

Find a basis of V.

V=
$$\int a(x+1) + bx(x+1) + Cx^{2}(x+1) : a,b,c \in \mathbb{R}$$

So (x+1), x(x+1), x(x+1)>

is a basis.

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	