國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

期末考 December 25, 2024 **Final Exam**

姓名 Name : Solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 4 & 4 \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
It is known that R is an echelon form of A.

(a) [1pt] Find at least two elements in $\ker(A)$.

e.g. $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

(b) [1pt] Find at least two elements **not** in $\ker(A)$.

e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(c) [1pt] Find a basis of $\text{Row}(A)$.

$\langle (1,1,1,0,0), (0,0,0,1,1) \rangle$

(d) [1pt] Find a basis of $\ker(A)$.

$\left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle$

(e) [1pt] Find a basis of $\text{Col}(A)$.

$\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\rangle$

2. Let $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the vector space of polynomial of degree at most 2. Let

$$B_1 = \langle 1, x, x^2 \rangle,\ B_2 = \langle 1, (x+1), (x+1)^2 \rangle,\ B_3 = \langle f_1(x), f_2(x), f_3(x) \rangle,$$

where

$$f_1(x) = \frac{(x-3)(x-5)}{(1-3)(1-5)},\ f_2(x) = \frac{(x-1)(x-5)}{(3-1)(3-5)},\ f_3(x) = \frac{(x-1)(x-3)}{(5-1)(5-3)}.$$

It is known that each of B_1, B_2, B_3 is a basis of $\mathcal{P}_{\cancel{3}}$ 2. Let

$$p(x) = 1 + 2x + 3x^2 \text{ and } \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) [1pt] Find $\mathrm{Rep}_{B_1}(p(x))$.

$p(x) = 1 \cdot 1 + 2 \cdot x + 3 \cdot x^2$, so $\mathrm{Rep}_{B_1}(p(x)) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) [1pt] Find $\mathrm{Rep}_{B_2}(p(x))$.

$p(x) = 2 - 4(x+1) + 3 \cdot (x+1)^2$, so $\mathrm{Rep}_{B_2}(p(x)) = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$.

(c) [1pt] Find $\mathrm{Rep}_{B_3}(p(x))$.

Note that $p(x) = p(1) \cdot f_1(x) + p(3) \cdot f_2(x) + p(5) \cdot f_3(x)$

$= 6 f_1 + 34 f_2 + 81 f_3$,

so $\mathrm{Rep}_{B_3}(p(x)) = \begin{bmatrix} 6 \\ 34 \\ 86 \end{bmatrix}$.

(d) [1pt] Find a polynomial $q(x)$ such that $\mathrm{Rep}_{B_2}(q(x)) = \mathbf{u}$.

$q(x) = 1 + 2(x+1) + 3(x+1)^2$

(e) [1pt] Find a polynomial $q(x)$ such that $\mathrm{Rep}_{B_3}(q(x)) = \mathbf{u}$.

$q(x) = 1 f_1(x) + 2 f_2(x) + 3 f_3(x)$

3. Let $\mathcal{M}_{2\times 2}$ be the vector space of all 2×2 real matrices. Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

be a subspace of $\mathcal{M}_{2\times 2}$.

(a) [1pt] Find a basis of $\mathcal{M}_{2\times 2}$.

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

(b) [1pt] Find at least two elements in V.

e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

(c) [3pt] Find a basis of V. Hint: Write down the equations for a, b, c, d.

~~Solve~~ Equate each entry in

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So $\begin{cases} a + c = 0 \\ b + d = 0 \end{cases}$

$\Rightarrow$ Parametrize it with c, d and get

$$V = \left\{ \begin{bmatrix} -c & -d \\ c & d \end{bmatrix} : c, d \in \mathbb{R} \right\}$$

So the basis is $\left\langle \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \right\rangle$

4. [5pt] Mathematical essay: Write a few paragraphs to introduce *dimension.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. Let $B = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \rangle$ be a basis of a vector space V. Suppose $\mathbf{p} = \mathbf{u}_1 + \mathbf{u}_4$. Let

$$B_1 = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{p}, \mathbf{u}_4 \rangle, \text{ and}$$
$$B_2 = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{p} \rangle.$$

(a) [extra 1pt] Explain why B_1 is not a basis of V.

~~So~~ B_1 is not independent since $-\vec{u}_1 - \vec{u}_4 + \vec{p} = \vec{0}$.

(b) [extra 2pt] Show that B_2 is independent.

Suppose $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{p} = \vec{0}$

Then $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4(\vec{u}_1 + \vec{u}_4) = \vec{0}$

and $(c_1+c_4)\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4 = \vec{0}$.

Since B is a basis, $\begin{cases} c_1 + c_4 = 0 \\ c_2 = 0 \\ c_3 = 0 \\ c_4 = 0 \end{cases} \Rightarrow c_1 = 0$.

So B_2 is independent.

(c) [extra 2pt] Show that $\text{span}(B_2) = V$.

Let $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4$ be an element in V.

Then $\vec{v} = (c_1 - c_4)\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{p} \in \text{span}(B_2)$.

So $\text{span}(B_2) = V$.

6. [extra 2pt] Let

$$V = \{(x+1)(a+bx+cx^2) : a, b, c \in \mathbb{R}\}.$$

Find a basis of V.

$V = \{ a(x+1) + bx(x+1) + cx^2(x+1) : a, b, c \in \mathbb{R} \}$

So $\langle (x+1), x(x+1), x^2(x+1) \rangle$

is a basis.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	