國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

第二次期中考 November 20, 2024 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let
$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}.$$

(a) [1pt] List any four elements in V.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(b) [1pt] Is V closed under addition? Provide your reasons.

No, e.g., $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -5 \end{bmatrix}$ are in V

but $\begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \notin V$.

(c) [1pt] Is V closed under scalar multiplication? Provide your reasons.

Yes, if $\begin{bmatrix} x \\ y \end{bmatrix}$ has $xy \geq 0$

then $\begin{bmatrix} kx \\ ky \end{bmatrix}$ also has $(kx)(ky) = k^2 \cdot xy \geq 0$

for any $k \in \mathbb{R}$.

(d) [1pt] Does V contain a zero vector? If yes, what is the zero vector in V?

Yes. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(e) [1pt] Find the additive inverse of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$\begin{bmatrix} -2 \\ -3 \end{bmatrix}$

2. Let $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ with

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) [1pt] Find an element in $[S]$.

$$u_1 + u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) [1pt] Find an element not in $[S]$.

Any element $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in [S]$ must have $x_1 = x_4$,

so $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \notin S$.

(c) [1pt] Prove, by definition, that $[S]$ is closed under addition.

Note that $[S] = \{ a_1\vec{u}_1 + a_2\vec{u}_2 : a_1, a_2 \in \mathbb{R} \}$.

Suppose $a_1\vec{u}_1 + a_2\vec{u}_2 \in [S]$ and $b_1\vec{u}_1 + b_2\vec{u}_2 \in [S]$.

Then their sum is $(a_1+b_1)\vec{u}_1 + (a_2+b_2)\vec{u}_2 \in [S]$.

(d) [1pt] Prove, by definition, that $[S]$ is closed scalar multiplication.

Suppose $a_1\vec{u}_1 + a_2\vec{u}_2 \in [S]$ and $k \in \mathbb{R}$.

Then $k(a_1\vec{u}_1 + a_2\vec{u}_2) = (ka_1)\vec{u}_1 + (ka_2)\vec{u}_2 \in [S]$

(e) [1pt] Is S an independent set? Provide your reasons.

Yes. Suppose $c_1\vec{u}_1 + c_2\vec{u}_2 = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

for some $c_1, c_2 \in \mathbb{R}$.

Then $\begin{cases} c_1 + 0c_2 = 0 \\ 0 + c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$.

3. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$ be the columns of

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 4 & 1 & 1 & 2 \\ 1 & 3 & 6 & 2 & 1 & 3 \\ 1 & 4 & 8 & 1 & 2 & 3 \end{bmatrix}$$

(a) [2pt] Find two different nontrivial linear relations in S.

$$A \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 & 0 & 1 \\ 0 & 3 & 6 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

$A\begin{bmatrix} c_1 \\ \vdots \\ c_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has solutions $\left\{ c_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} : c_3, c_6 \in \mathbb{R} \right\}$.

e.g. Pick $\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{-2\vec{u}_2 + \vec{u}_3 = \vec{0}}$

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \underline{-\vec{u}_4 - \vec{u}_5 + \vec{u}_6 = \vec{0}}$

(b) [3pt] Find a subset $T \subset S$ such that T is linearly independent and $[T] = [S]$. Provide your reasons.

By $-2\vec{u}_2 + \vec{u}_3 = \vec{0}$. Removing $\vec{u}_3$ will not change the span.

By $-\vec{u}_4 - \vec{u}_5 + \vec{u}_6 = \vec{0}$ Removing $\vec{u}_6$ will not change the span.

Choose $T = \{\vec{u}_1, \vec{u}_2, \vec{u}_4, \vec{u}_5\}$.

There is no nontrivial solution above with $c_3 = c_6 = 0$.

So $[T] = [S]$ and T is indep.

4. [5pt] Mathematical essay: Write a few paragraphs to introduce *linear independence.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. [extra 2pt] Let $S = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be a set of vectors. Suppose S is linearly independent. Show that $T = \{\mathbf{x}, \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} + \mathbf{z}\}$ is also linearly independent.

Suppose $c_1 \vec{x} + c_2 (\vec{x} + \vec{y}) + c_3 (\vec{x} + \vec{y} + \vec{z}) = \vec{0}$

Then $(c_1 + c_2 + c_3)\vec{x} + (c_2 + c_3)\vec{y} + c_3 \vec{z} = \vec{0}$.

Since S is indep, we have

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \\ c_3 = 0 \end{cases},$$

which leads to $c_1 = c_2 = c_3 = 0$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	