姓名 Name ：_____

學號 Student ID # ：_____

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **6 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.

- Using the calculator is not allowed (and not necessary) for this exam.

- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.

- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.

- Please answer the problems in English.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear function, $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of $\mathbb{R}^3$, and $\beta = \{\mathbf{u}_1, \mathbf{u}_2\}$ a basis of $\mathbb{R}^2$ such that

$$[f]_\alpha^\beta = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_2 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}.$$

(a) [1pt] Find $f(\mathbf{v}_2)$.

(b) [1pt] Find $f(\mathbf{v}_3)$.

(c) [1pt] Find $f(\mathbf{v}_1 + \mathbf{v}_2)$.

(d) [1pt] Find $[\mathbf{v}_1 + \mathbf{v}_3]_\alpha$.

(e) [1pt] Find $f\left(\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} \right)$.

2. Let
$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}.$$

(a) [2pt] For the optimization problem
$$\begin{aligned} \max \quad & \mathbf{x}^\top A\mathbf{x}, \\ \text{subject to} \quad & \|\mathbf{x}\| = 1, \end{aligned}$$
find the maximum value and the $\mathbf{x}$ that achieves it.

(b) [1pt] Let
$$D = \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}.$$
Find a vector $\mathbf{x}$ with $\|\mathbf{x}\| = 1$ such that $\mathbf{x}^\top D\mathbf{x} = 0$.

(c) [2pt] Find a vector $\mathbf{x}$ with $\|\mathbf{x}\| = 1$ such that $\mathbf{x}^\top A\mathbf{x} = 0$.

3. Let

$$A = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

(a) [4pt] Find the spectral decomposition

$$A = \sum_{j=1}^{q} \mu_j P_j.$$

(b) [1pt] Explain what does the action $\mathbf{x} \mapsto A\mathbf{x}$ do.

4. [5pt] Mathematical essay: Write a few paragraphs to introduce *linear regression.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. [extra 5pt] Let
$$A = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 1 & 1 & 2 & 2 \end{bmatrix}.$$

Find the singular value decomposition
$$A = U\Sigma V^\top.$$

6. [extra 2pt] Consider the 10×10 multiplication table as a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 & 40 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 & 60 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 & 70 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 & 80 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 & 90 \\ 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 & 100 \end{bmatrix}.$$

Find all the eigenvalues of A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	