國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

期末考 June 3, 2024 **Final Exam**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 **5 pages** of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear function, $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of $\mathbb{R}^3$, and $\beta = \{\mathbf{u}_1, \mathbf{u}_2\}$ a basis of $\mathbb{R}^2$ such that

$$[f]_\alpha^\beta = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_2 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}.$$

(a) [1pt] Find $f(\mathbf{v}_2)$.

$f(\vec{v}_2) = 2 \cdot \vec{u}_2 = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$

(b) [1pt] Find $f(\mathbf{v}_3)$.

$f(\vec{v}_3) = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(c) [1pt] Find $f(\mathbf{v}_1 + \mathbf{v}_2)$.

$f(\vec{v}_1 + \vec{v}_2) = 5\vec{u}_1 + 2\vec{u}_2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 12 \\ 14 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \end{bmatrix}$

(d) [1pt] Find $[\mathbf{v}_1 + \mathbf{v}_3]_\alpha$.

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(e) [1pt] Find $f\left(\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}\right)$.

$\left[\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}\right]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ So $f\left(\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}\right) = f(\vec{v}_1 + \vec{v}_3) = 5\vec{u}_1 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

2. Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}.$$

(a) [2pt] For the optimization problem

$$\begin{aligned} \max \quad & \mathbf{x}^\top A \mathbf{x}, \\ \text{subject to} \quad & \|\mathbf{x}\| = 1, \end{aligned}$$

find the maximum value and the $\mathbf{x}$ that achieves it.

A has char poly $x^2 - 4x - 12 = (x-6)(x+2)$.

So eigvals: $-2 < 6$.

eigvecs: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

max $= \underline{6}$.

$\vec{x}$ is the normalized vector of $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

(b) [1pt] Let

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}.$$

Find a vector $\mathbf{x}$ with $\|\mathbf{x}\| = \cancel{0}\ 1$ such that $\mathbf{x}^\top D \mathbf{x} = 0$.

$[a\ b]\, D \begin{bmatrix} a \\ b \end{bmatrix} = a^2 \cdot (-2) + b^2 \cdot 6$

With $a^2 + b^2 = 1$, choose $a^2 = \frac{3}{4}$, $b^2 = \frac{1}{4}$.

So $\vec{x}$ could be $\underline{\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}}$

(c) [2pt] Find a vector $\mathbf{x}$ with $\|\mathbf{x}\| = \cancel{0}\ 1$ such that $\mathbf{x}^\top A \mathbf{x} = 0$.

Choose $\vec{x} = \frac{\sqrt{3}}{2} \cdot \left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) + \frac{1}{2} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

$$= \underline{\begin{bmatrix} \frac{-\sqrt{3}+1}{2\sqrt{2}} \\ \frac{\sqrt{3}+1}{2\sqrt{2}} \end{bmatrix}}$$

3. Let

$$A = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

$3 \quad -6 \quad -6$

$-4 \quad -4 \quad -4 \quad -8 \quad +1 \quad -8$

(a) [4pt] Find the spectral decomposition

$$A = \sum_{j=1}^{q} \mu_j P_j.$$

It's easier to check $B = 3A = \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & -1 \end{bmatrix}$ instead.

~~charpoly : x~~

charpoly of B $: -x^3 + 3x^2 + 9x - 27$
"
charpoly of A $= -(x-3)(x^2 - 9) = -(x-3)(x+3)(x+3)$.

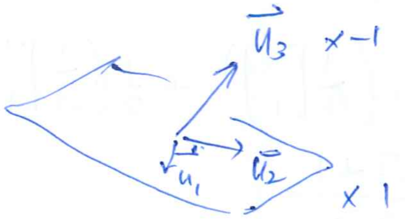
eigvals of B : $3, 3, -3$.

eigvals of A : $\underbrace{1, 1}_{\mu_1}, \underbrace{-1}_{\mu_2}$.

eigvecs of B
"
eigvecs of A : $\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\vec{u}_1}, \underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\vec{u}_2}, \underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\vec{u}_3}$

$$A = \mu_1(\vec{u}_1\vec{u}_1^T + \vec{u}_2\vec{u}_2^T) + \mu_2\vec{u}_3\vec{u}_3^T = 1 \cdot \begin{bmatrix} 5/6 & -1/6 & -2/6 \\ -1/6 & 5/6 & -2/6 \\ -2/6 & -2/6 & 2/6 \end{bmatrix} + (-1)\begin{bmatrix} 1/6 & 1/6 & 2/6 \\ 1/6 & 1/6 & 2/6 \\ 2/6 & 2/6 & 4/6 \end{bmatrix}$$

(b) [1pt] Explain what does the action $\mathbf{x} \mapsto A\mathbf{x}$ do.



It's a reflection w.r.t the plane span$\{\vec{u}_1, \vec{u}_2\}$.

4. [5pt] Mathematical essay: Write a few paragraphs to introduce *linear regression.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. [extra 5pt] Let

$$A = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 1 & 1 & 2 & 2 \end{bmatrix}.$$

Find the singular value decomposition

$$A = U\Sigma V^\top.$$

$$A^T A = \begin{bmatrix} 2 & 2 & & \\ 2 & 2 & & 0 \\ & & 8 & 8 \\ 0 & & 8 & 8 \end{bmatrix} \to \text{eigvals: } 4, 0, 16, 0 \quad (16 \geq 4 \geq 0 \geq 0)$$

eigvecs: $\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}_{\vec{v}_2}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}}_{\vec{v}_3}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}_{\vec{v}_1}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}}_{\vec{v}_4}$.

Singular values = $\sqrt{\text{eigvals}}$ = $\underset{\sigma_1}{4}$, $\underset{\sigma_2}{2}$.

$$A\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-4\\4\end{bmatrix} = 4\underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right)}_{\vec{u}_1}$$

$$A\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}2\\2\end{bmatrix} = 2\underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)}_{\vec{u}_2}$$

$$A = \underbrace{\begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ - & \vec{v}_3^T & - \\ - & \vec{v}_4^T & - \end{bmatrix}}_{V^T}$$

6. [extra 2pt] Consider the 10×10 multiplication table as a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 & 40 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 & 60 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 & 70 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 & 80 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 & 90 \\ 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 & 100 \end{bmatrix}.$$

$1^2+2^2+\cdots+10^2$

$\parallel$

$\frac{n(n+1)(2n+1)}{6}$

$\frac{\cancel{10}^{5}\cdot 11\cdot \cancel{21}^{7}}{\cancel{6}} = 385$

Find all the eigenvalues of A.

Let $M = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 10 \end{bmatrix}$. Then $A = MM^T$.

It's nonzero eigvals are the same with ~~MM^T~~

$M^TM = [1^2+2^2+\cdots+10^2] = 385$

eigvals of A : $385,\ 0^{(9)}$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	