國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

期末考 June 3, 2024 **Final Exam**

姓名 Name : Solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 ~~**5**~~ **pages** of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear function, $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of $\mathbb{R}^3$, and $\beta = \{\mathbf{u}_1, \mathbf{u}_2\}$ a basis of $\mathbb{R}^2$ such that

$$[f]_\alpha^\beta = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_2 = \begin{bmatrix} 8 \\ 9 \end{bmatrix}.$$

(a) [1pt] Find $f(\mathbf{v}_1)$.

$$\underline{f(\vec{v}_1) = 4\vec{u}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}}$$

(b) [1pt] Find $f(\mathbf{v}_3)$.

$$\underline{f(\vec{v}_3) = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

(c) [1pt] Find $f(\mathbf{v}_1 + \mathbf{v}_2)$.

$$f(\vec{v}_1 + \vec{v}_2) = \underline{4\vec{u}_1 + 3\vec{u}_2} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 24 \\ 27 \end{bmatrix} = \underline{\begin{bmatrix} 28 \\ 31 \end{bmatrix}}$$

(d) [1pt] Find $[\mathbf{v}_2 + \mathbf{v}_3]_\alpha$.

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(e) [1pt] Find $f\left(\begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}\right)$.

$$f\left(\begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}\right) = f(\vec{v}_2 + \vec{v}_3) = \underline{3\vec{u}_2 = \begin{bmatrix} 24 \\ 27 \end{bmatrix}}$$

2. Let

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}.$$

(a) [2pt] For the optimization problem

$$\begin{array}{ll} \max & \mathbf{x}^\top A\mathbf{x}, \\ \text{subject to} & \|\mathbf{x}\| = 1, \end{array}$$

find the maximum value and the $\mathbf{x}$ that achieves it.

charpoly : $x^2 - 3x - 4 = (x-4)(x+1)$

eigvals : $-1 < 4$

normalized eigvecs : $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

max : 4

$\vec{x}$: $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) [1pt] Let

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Find a vector $\mathbf{x}$ with $\|\mathbf{x}\| = \cancel{0}\ 1$ such that $\mathbf{x}^\top D\mathbf{x} = 0$.

$[a\ b]\, D \begin{bmatrix} a \\ b \end{bmatrix} = a^2 \cdot (-1) + b^2 \cdot 4$

With $a^2 + b^2 = 1$, choose $a^2 = \frac{4}{5}$, $b^2 = \frac{1}{5}$.

Then $\vec{x}$ could be $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

(c) [2pt] Find a vector $\mathbf{x}$ with $\|\mathbf{x}\| = \cancel{0}\ 1$ such that $\mathbf{x}^\top A\mathbf{x} = 0$.

Choose $\vec{x} = \frac{2}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) + \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

$= \begin{bmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix}$.

3. Let

$$A = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

(a) [4pt] Find the spectral decomposition

$$A = \sum_{j=1}^{q} \mu_j P_j.$$

See ver. A

(b) [1pt] Explain what does the action $\mathbf{x} \mapsto A\mathbf{x}$ do.

4. [5pt] Mathematical essay: Write a few paragraphs to introduce *linear regression.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. [extra 5pt] Let

$$A = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 1 & 1 & 2 & 2 \end{bmatrix}.$$

Find the singular value decomposition

$$A = U\Sigma V^\top.$$

See ver. A.

6. [extra 2pt] Consider the 10×10 multiplication table as a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 & 40 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 & 60 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 & 70 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 & 80 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 & 90 \\ 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 & 100 \end{bmatrix}.$$

Find all the eigenvalues of A.

See Ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	