國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第一次期中考 March 25, 2024 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

| | |
|---|---|
| Lecturer: | Jephian Lin 林晉宏 |
| Contents: | cover page, **5 pages** of questions, score page at the end |
| To be answered: | on the test paper |
| Duration: | **110 minutes** |
| Total points: | **20 points** + 2 extra points |

**Do not open this packet until instructed to do so.**

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Let

$$A = \begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{y} & - \\ - & \mathbf{z} & - \end{bmatrix}, \quad B = \begin{bmatrix} - & \mathbf{y} & - \\ - & \mathbf{z} & - \\ - & \mathbf{x} & - \end{bmatrix}, \quad C = \begin{bmatrix} - & 3\mathbf{x} & - \\ - & 4\mathbf{y} & - \\ - & 5\mathbf{z} & - \end{bmatrix},$$

$$D = \begin{bmatrix} - & \mathbf{x} & - \\ - & 2\mathbf{x}+\mathbf{y} & - \\ - & 5\mathbf{x}+\mathbf{z} & - \end{bmatrix}, \quad E = \begin{bmatrix} - & \mathbf{x}+3\mathbf{y} & - \\ - & \mathbf{y}+3\mathbf{z} & - \\ - & \mathbf{z}+3\mathbf{x} & - \end{bmatrix}, \quad F = \begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{y}+\mathbf{z} & - \\ - & \mathbf{y}+\mathbf{z} & - \end{bmatrix}.$$

Let $\det(A) = \Delta$.

(a) [1pt] Find $\det(B)$. Provide your reasons.

$A \xrightarrow[\rho_2 \leftrightarrow \rho_1]{\rho_3 \leftrightarrow \rho_2} B$, so $\det(B) = (-1)^2 \cdot \det(A) = \underline{\underline{\Delta}}$.

(b) [1pt] Find $\det(C)$. Provide your reasons.

$A \xrightarrow{\rho_1 : \times 3} \xrightarrow{\rho_2 : \times 4} \xrightarrow{\rho_3 : \times 5} B$, so $\det(B) = \cancel{2}3 \cdot 4 \cdot 5 \cdot \det(A)$

$= \underline{\underline{60\Delta}}$

(c) [1pt] Find $\det(D)$. Provide your reasons.

$A \xrightarrow{\rho_2 : +2\rho_1} \xrightarrow{\rho_3 : +5\rho_1} D$, so $\det(B) = \det(A) = \underline{\underline{\Delta}}$

(d) [1pt] Find $\det(E)$. Provide your reasons.

$E = \underbrace{\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{bmatrix}}_{M} \cdot A$, so $\det(E) = \cancel{\det[\ ]} \det(M) \cdot \det(A)$

$= \underline{\underline{28\Delta}}$

$\det(M) = 1 + 27 = 28$

(e) [1pt] Find $\det(F)$. Provide your reasons.

$\det(F) = 0$ since $F$ contains repeated rows.

2. Let

$$A = \begin{bmatrix} a & b & c & d \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}.$$

(a) [4pt] Find $\det(A)$ in terms of variables $a$, $b$, $c$, and $d$.

By Laplace expansion,

$$\det(A) = a \cdot \underbrace{\det\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}}_{2-4-1} - b \cdot \underbrace{\det\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{1+1-2-2} + c \cdot \underbrace{\det\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}}_{1+8-2-2} - d \cdot \underbrace{\det\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}}_{4-1-2}.$$

$$= \underline{-3a + 2b + 5c - d}.$$

(b) [1pt] Find some **nonzero** values of $a$, $b$, $c$ and $d$ such that $\det(A)$ is zero.

Choose any nonzero numbers to make $-3a + 2b + 5c - d = 0$.

For example, $a = 2$, $b = 1$, $c = 1$, $d = 1$.

3. [5pt] Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 1 & -6 \\ -16 & -3 & 23 \end{bmatrix}.$$

Write $A$ as a product of elementary matrices.

$$A \xrightarrow[\rho_3 : +16\rho_1]{\rho_2 : -4\rho_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 7 \end{bmatrix} \xrightarrow{\rho_3 : +3\rho_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\rho_2 : +2\rho_3]{\rho_1 : +1\rho_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rho_3 : -16\rho_1$

$\rho_2 : +4\rho_1$

$\rho_3 : -3\rho_2$

$\rho_2 : -2\rho_3$

$\rho_1 : -1\rho_3$

So.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -16 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. [5pt] Mathematical essay: Write a few paragraphs to introduce *permutation expansion.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

be a $10 \times 10$ matrix. Find $\det(A)$.

$$A \xrightarrow[i=1\sim 9]{\rho_{10}:-\rho_i} \begin{bmatrix} 1 & & O & 1 \\ & \ddots & & \vdots \\ O & & 1 & 1 \\ & & & -8 \end{bmatrix}$$

$\det(A) = \underline{\underline{-8}}$.

| Page | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 2 | |
| Total | 20 (+2) | |