國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第二次期中考 April 29, 2024 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function and $\beta = \{\mathbf{u}_1, \mathbf{u}_2\}$ a basis of $\mathbb{R}^2$ such that

$$[f]_\beta^\beta = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_2 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}.$$

Let $\mathcal{E}_2$ be the standard basis of $\mathbb{R}^2$.

(a) [1pt] Find $f(\mathbf{u}_2)$.

$$f(\vec{u}_2) = 0\vec{u}_1 + 5\vec{u}_2 = \begin{bmatrix} 30 \\ 35 \end{bmatrix}$$

(b) [1pt] Find $f(\mathbf{u}_1 + \mathbf{u}_2)$.

$$f(\vec{u}_1 + \vec{u}_2) = f(\vec{u}_1) + f(\vec{u}_2)$$
$$= 2\vec{u}_1 + 5\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 30 \\ 35 \end{bmatrix} = \begin{bmatrix} 32 \\ 37 \end{bmatrix}$$

(c) [1pt] Find $[\mathrm{id}]_\beta^{\mathcal{E}_2}$.

$$[\mathrm{id}]_\beta^{\mathcal{E}_2} = \begin{bmatrix} | & | \\ [\vec{u}_1]_{\mathcal{E}_2} & [\vec{u}_2]_{\mathcal{E}_2} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 7 \end{bmatrix}$$

(d) [1pt] Find $[\mathrm{id}]_{\mathcal{E}_2}^\beta$.

$$[\mathrm{id}]_{\mathcal{E}_2}^\beta = \left([\mathrm{id}]_\beta^{\mathcal{E}_2}\right)^{-1} = \begin{bmatrix} 7 & -6 \\ -1 & 1 \end{bmatrix}$$

(e) [1pt] Find $f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$.

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = -\begin{bmatrix} 12 \\ 12 \end{bmatrix} + \begin{bmatrix} 12 \\ 14 \end{bmatrix} = -12\vec{u}_1 + 2\vec{u}_2$$

$$f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = f(-12\vec{u}_1) + f(2\vec{u}_2)$$
$$= -24\vec{u}_1 + 10\vec{u}_2$$
$$= \begin{bmatrix} -24 \\ -24 \end{bmatrix} + \begin{bmatrix} 60 \\ 70 \end{bmatrix} = \begin{bmatrix} 36 \\ 46 \end{bmatrix}.$$

2. [3pt] Let

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

Find the characteristic polynomial of A.

$P_A(x) = (-x)^5 + s_1(-x)^4 + s_2(-x)^3 + s_3(-x)^2 + s_4(-x) + s_5$.

$s_1 = tr(A) = 3+0+0+0+3 = \underline{\underline{6}}$.

$s_2 = $ sum of $\det(2\times 2) = \det\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \times 3 + \det\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \times 3$

$+ 0 \times 4$

$= \underline{\underline{-6}}$.

$\underline{\underline{s_3 = s_4 = s_5 = 0}}$ since all $k\times k$ ~~princip~~ submatx contains repeated rows. for $k \geq 3$

3. [2pt] Let

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}.$$

Diagonalize A or provide reasons showing A is not diagonalizable.

$P_A(x) = x^2 - 0x + 0 = x^2$

$\lambda = 0, 0$. $\Rightarrow a.m_A(\lambda) = 2$.

For $\lambda = 0$,

$A - \lambda I = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$.

So $g.m_A(\lambda) = \cancel{\#}$ null $(A - \lambda I) = 1$.

$\underline{\underline{\text{Not diagonalizable}}}$, since $a.m(0) \neq g.m(0)$.

4. ~~[5pt]~~ Let

$$A = \begin{bmatrix} 6 & -8 & -4 \\ -1 & 1 & 1 \\ 10 & -14 & -8 \end{bmatrix}.$$

(a) [2pt] Find the characteristic polynomial and all eigenvalues of A.

$$P_A(x) = (-x)^3 + s_1(-x)^2 + s_2(-x) + s_3$$

$$s_1 = tr(A) = 6+1+(-8) = -1$$

$$s_2 = \det\begin{bmatrix} 6 & -8 \\ -1 & 1 \end{bmatrix} + \det\begin{bmatrix} 6 & -4 \\ 10 & -8 \end{bmatrix} + \det\begin{bmatrix} 1 & 1 \\ -14 & -8 \end{bmatrix}$$

$$= (-2) + (-8) + 6 = -4.$$

$$s_3 = \det(A) = -48 - 80 - 56 + 40 + 64 + 84 = 4$$

$$P_A(x) = \underline{-x^3 - x^2 + 4x + 4} = -x^2(x+1) + 4(x+1) = \cancel{-x^2(x+1}$$

$$= -(x+2)(x-2)(x+1) \quad \cancel{=(x^2-4)}$$

eigvals = $\underline{2, -2, -1}$.

(b) [3pt] Find a basis β such that $[f_A]_\beta^\beta$ is diagonal.

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 4 & -8 & -4 \\ -1 & -1 & 1 \\ 10 & -14 & -10 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\ker(A - 2I) = span\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{u}_1} \right\}.$$

For $\lambda = -2$,

$$A + 2I = \begin{bmatrix} 8 & -8 & -4 \\ -1 & 3 & 1 \\ 10 & -14 & -6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & -1 \\ 2 & -2 & -1 \\ 5 & -7 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \cancel{-3}\,1 & \cancel{-1}\,0 \\ 0 & 4 & 1 \\ 0 & \cancel{8}\,0 & \cancel{2}\,0 \end{bmatrix}$$

$$\ker(A + 2I) = span\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}}_{\vec{u}_2} \right\}.$$

For $\lambda = -1$

$$A + I = \begin{bmatrix} 7 & -8 & -4 \\ -1 & 2 & 1 \\ 10 & -14 & -7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \cancel{-2}\,0 & \cancel{-1}\,0 \\ 0 & \cancel{6}\,2 & \cancel{3}\,1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A + I) = span\left\{ \underbrace{\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}}_{\vec{u}_3} \right\}.$$

$$\underline{\beta = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}}.$$

5. [5pt] Mathematical essay: Write a few paragraphs to introduce *diagonalization.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

6. [extra 2pt] Solve a_n in the recurrence relation

$$\begin{cases} a_{n+2} = a_{n+1} + 2a_n, \\ a_0 = 0,\ a_1 = 1. \end{cases}$$

n	0	1	2	3	4	
a_n	0	1	1	3	5	...

$$\begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$$

$$\vec{v}_{n+1} = A \cdot \vec{v}_n \Rightarrow \vec{v}_n = A^n \cdot \vec{v}_0$$

Diagonalize A:

$p_A(x) = x^2 - x - 2 = (x-2)(x+1) \Rightarrow$ eigvals $= 2, -1$.

$\lambda = 2$.

$A - 2I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \Rightarrow \ker(A-2I) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{u}_1} \right\}$

$\lambda = -1$

$A + I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow \ker(A+I) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{u_2} \right\}$.

Let $Q = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Then $Q^{-1} = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} / -3$

$a_n = $ 1st entry of $\vec{v}_n$

$= $ 1st entry of $QD^nQ^{-1} \cdot \vec{v}_0$ $\swarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= -\frac{1}{3} [1 \ 1] \begin{bmatrix} 2^n & \\ & (-1)^n \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= -\frac{1}{3}((-1)^n - 2^n)$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	