國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第二次期中考 April 29, 2024 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

– Enter your **Name** and **Student ID #** before you start.

– Using the calculator is not allowed (and not necessary) for this exam.

– Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.

– Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.

– Please answer the problems in English.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function and $\beta = \{\mathbf{u}_1, \mathbf{u}_2\}$ a basis of $\mathbb{R}^2$ such that
$$[f]_\beta^\beta = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_2 = \begin{bmatrix} 8 \\ 9 \end{bmatrix}.$$
Let $\mathcal{E}_2$ be the standard basis of $\mathbb{R}^2$.

(a) [1pt] Find $f(\mathbf{u}_2)$.

$$f(\vec{u}_2) = 4\vec{u}_2 = \begin{bmatrix} 32 \\ 36 \end{bmatrix}$$

(b) [1pt] Find $f(\mathbf{u}_1 + \mathbf{u}_2)$.

$$f(\vec{u}_1 + \vec{u}_2) = f(\vec{u}_1) + f(\vec{u}_2)$$
$$= 3\vec{u}_1 + 4\vec{u}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} \cancel{32} \\ 36 \end{bmatrix} = \begin{bmatrix} 35 \\ 39 \end{bmatrix}$$

(c) [1pt] Find $[\mathrm{id}]_\beta^{\mathcal{E}_2}$.

$$[id]_\beta^{\mathcal{E}_2} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 1 & 9 \end{bmatrix}$$

(d) [1pt] Find $[\mathrm{id}]_{\mathcal{E}_2}^\beta$.

$$[id]_{\mathcal{E}_2}^\beta = \left([id]_\beta^{\mathcal{E}_2}\right)^{-1} = \begin{bmatrix} 9 & -8 \\ -1 & 1 \end{bmatrix}$$

(e) [1pt] Find $f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$.

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = -\begin{bmatrix} 16 \\ 16 \end{bmatrix} + \begin{bmatrix} 16 \\ 18 \end{bmatrix} = -16\vec{u}_1 + 2\vec{u}_2$$

$$f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \cancel{0} - 48\vec{u}_1 + 8\vec{u}_2$$
$$= \begin{bmatrix} -48 \\ -48 \end{bmatrix} + \begin{bmatrix} 64 \\ 72 \end{bmatrix} = \begin{bmatrix} 16 \\ 24 \end{bmatrix}$$

2. [3pt] Let

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

Find the characteristic polynomial of A.

$P_A(x) = (-x)^5 + s_1(-x)^4 + s_2(-x)^3 + s_3(-x)^2 + s_4(-x) + s_5.$

$s_1 = tr(A) = 2+0+0+0+2 = \underline{\underline{4}}.$

$s_2 = \det\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \times 3 + \det\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \times 3 + 0 \times 4 = \underline{\underline{-6}}$

$\underline{\underline{s_3 = s_4 = s_5 = 0}}$ since all $k \times k$ submtx has repeated rows for $k \geq 3$.

3. [2pt] Let

$$A = \begin{bmatrix} 3 & 1 \\ -6 & -3 \end{bmatrix}.$$

Diagonalize A or provide reasons showing A is not diagonalizable.

$P_A(x) = x^2 - 3 \Rightarrow$ eigvals $= \sqrt{3}, -\sqrt{3}$

all eigvals distinct $\Rightarrow$ diagonalizable

4. [5pt] Let

$$A = \begin{bmatrix} 8 & 2 & -4 \\ -9 & -2 & 5 \\ 7 & 2 & -3 \end{bmatrix}.$$

(a) [2pt] Find the characteristic polynomial and all eigenvalues of A.

$$P_A(x) = (-x)^3 + s_1(-x)^2 + s_2(-x) + s_3$$

$$s_1 = \mathrm{tr}(A) = 8-2-3 = 3.$$

$$s_2 = \det\begin{bmatrix} 8 & 2 \\ -9 & -2 \end{bmatrix} + \det\begin{bmatrix} -2 & 5 \\ 2 & -3 \end{bmatrix} + \det\begin{bmatrix} 8 & -4 \\ 7 & -3 \end{bmatrix}$$

$$= 2 - 4 + 4 = 2$$

$$s_3 = \det(A) = \underbrace{48 + 70 + 72}_{190} \underbrace{- 56 - 54 - 80}_{-190} = \cancel{96}\ 0.$$

$$P_A(x) = \underline{-x^3 + 3x^2 - 2x} = -x(x-1)(x-2)$$

$$\underline{\text{eigvals} = 2, 1, 0}$$

(b) [3pt] Find a basis β such that $[f_A]_\beta^\beta$ is diagonal.

$\lambda = 2$.

$$A - 2I = \begin{bmatrix} 6 & 2 & -4 \\ -9 & -4 & 5 \\ 7 & 2 & -5 \end{bmatrix} \rightsquigarrow \cancel{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 18 & 14 \\ 0 & -12 & -8 \end{bmatrix}} \rightsquigarrow \begin{bmatrix} 3 & 1 & -2 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A - 2I) = \mathrm{span}\left\{\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\vec{u}_1}\right\}.$$

$\lambda = 1$

$$A - I = \begin{bmatrix} 7 & 2 & -4 \\ -9 & -3 & 5 \\ 7 & 2 & -4 \end{bmatrix} \longrightarrow \cancel{\begin{bmatrix} 9 & 0 & -6 \\ 0 & 1 & 1 \\ 7 & 0 & 4 \end{bmatrix}} \rightsquigarrow \begin{bmatrix} 7 & 2 & -4 \\ -2 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A - I) = \mathrm{span}\left\{\underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{\vec{u}_2}\right\}.$$

$\lambda = 0$

$$A \rightsquigarrow \begin{bmatrix} 4 & 1 & -2 \\ -1 & 0 & 1 \\ 7 & 2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \ker(A - 0I) = \mathrm{span}\left\{\underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\vec{u}_3}\right\}$$

$$\underline{\beta = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}}.$$

5. [5pt] Mathematical essay: Write a few paragraphs to introduce *diagonalization.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

6. [extra 2pt] Solve a_n in the recurrence relation

$$\begin{cases} a_{n+2} = a_{n+1} + 2a_n, \\ a_0 = 0, \ a_1 = 1. \end{cases}$$

See Ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	