國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

MATH107A / GEAI1216A

數學導論（一） Introduction to Mathematics I

第二次期中考 November 28, 2025 **Midterm 2**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Give an example of "*d divides n*".

Answer: $3 \mid 12$

Reason: There exists an integer 4 s.t $12 = 3 \cdot 4$

2. [2pt] Prove that the sum of two even numbers is even, while the sum of two odd numbers is even.

Proof:

Let a and b even numbers. Then there exist integers m and n such that $a=2m$ and $b=2n$.
Then

$$a+b = 2m+2n = 2(m+n),$$

which is an even number.

Now let a and b odd numbers. Then there exist integers m and n such that $a=2m+1$ and $b=2n+1$.
Then

$$a+b = (2m+1)+(2n+1) = 2(m+n+1),$$

which is even.

Therefore, the sum of two numbers is even, while the sum of two odd numbers is even. ∎

3. [2pt] Prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for any positive integer n.

Proof:

Let $P(n)$ be the statement $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for any $n \in \mathbb{Z}^+$.

Step 1:

For $n=1$, we have

$$1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$$

So $P(1)$ holds.

Step 2:

Assume $P(m)$ is true, that is

$$P(m) = \sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}, \text{ for some } m \in \mathbb{Z}^+$$

We want to show that $P(m+1)$ also holds, that is

$$P(m+1) = \sum_{k=1}^{m+1} k^2 = \frac{(m+1)(m+2)(2m+3)}{6}, \text{ for some } m \in \mathbb{Z}^+$$

Now,

$$\begin{aligned}
\sum_{k=1}^{m+1} k^2 &= \sum_{k=1}^{m} k^2 + (m+1)^2 \\
&= \frac{m(m+1)(2m+1)}{6} + (m+1)^2 \\
&= (m+1)\left[\frac{m(2m+1)}{6} + (m+1)\right] \\
&= (m+1)\left[\frac{m(2m+1)+6m+6}{6}\right] \\
&= (m+1)\left[\frac{2m^2+7m+6}{6}\right] \\
&= \frac{(m+1)(m+2)(2m+3)}{6}, \text{ for some } m \in \mathbb{Z}^+
\end{aligned}$$

Thus, $P(m+1)$ also holds.

 Every sentence should be complete.

Therefore by PMI, $P(n)$ is true, $n \in \mathbb{Z}^+$ ∎

4. [1pt] Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. List all elements in the Cartesian product $X \times Y$.

Sol'n: Given $X = \{1,2,3\}$, $Y = \{a,b\}$

$X \times Y = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$.

5. [2pt] Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: $(\subseteq)$

Let $x \in A \cap (B \cup C)$.

Then $x \in A$ and $x \in (B \cup C)$.

So $x \in B$ or $x \in C$.

Thus $x \in A \cap B$ or $x \in A \cap C$.

Hence $x \in (A \cap B) \cup (A \cap C)$.

$(\supseteq)$

Let $x \in (A \cap B) \cup (A \cap C)$.

Then $x \in (A \cap B)$ or $x \in (A \cap C)$.

So $x \in A$ and also $x \in B$ or $x \in C$.

Thus $x \in A \cap (B \cup C)$.

Therefore the two sets are equal. ∎

6. [2pt] Prove or disprove: $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$. Here $\mathcal{P}(S)$ stands for the power set of S.

Answer: The statement is false.

Counterexample:

Let $A = \{1\}$ and $B = \{2\}$

Then

$A \cup B = \{1,2\}$ and

$P(A \cup B) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$.

But

$P(A) \cup P(B) = \{\phi, \{1\}\} \cup \{\phi, \{2\}\} = \{\phi, \{1\}, \{2\}\}$.

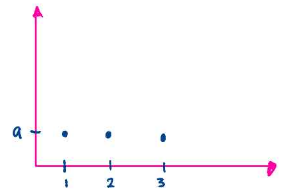
Since $\{1,2\} \in P(A \cup B)$ but $\{1,2\} \notin P(A) \cup P(B)$, the equality fails.

7. [1pt] Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. Consider the function $f : X \to Y$ defined by $f(1) = f(2) = f(3) = a$. What are the domain, the codomain, the graph, and the range of f?

Domain: $X = \{1,2,3\}$

Graph: $\{(1,a), (2,a), (3,a)\}$

Codomain: $Y = \{a,b\}$



Range: $\{a\}$

8. [2pt] Let $\mathbb{R}_+$ be the set of all positive real numbers. Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ by $g(x) = x^2$. Prove or disprove that g is injective (one-to-one).

Proof:

Let $x_1, x_2 \in \mathbb{R}_+$ and suppose

$$g(x_1) = g(x_2).$$

By definition of g, this means

$$x_1^2 = x_2^2.$$

Since both x_1 and x_2 are positive real numbers, taking positive square root gives

$$x_1 = x_2$$

Thus the function g is one-to-one. ∎

9. [2pt] For the same function g, prove or disprove that g is surjective (onto).

Proof:

Let $y \in \mathbb{R}_+$ be arbitrary.

Since $y > 0$, its square root $\sqrt{y}$ is a real number and is also positive.

Thus, $\sqrt{y} \in \mathbb{R}^+$.

Now,

$$g(\sqrt{y}) = (\sqrt{y})^2 = y.$$

Therefore, the positive real number $x = \sqrt{y}$ satisfies

$$g(x) = y.$$

Hence, g is surjective.

10. [1pt] Consder the set

$$Q = \{(p, q) : p, q \in \mathbb{Z} \text{ and } p, q > 0\}.$$

Define

$$E = \{\langle (p_1, q_1), (p_2, q_2)\rangle \in Q^2 : p_1q_2 = p_2q_1\}.$$

Find two elements from E.

Sol'n:

* $\langle (1,2),(2,4)\rangle$ because $1 \cdot 4 = 2 \cdot 2$
* $\langle (3,5),(6,10)\rangle$ because $3 \cdot 10 = 5 \cdot 6$.

11. [2pt] For the same set E, show that E is an equivalence.

Proof:

Reflexive:

Take any $(p,q) \in Q$,

$$pq = pq;$$

So $\langle (p,q),(p,q)\rangle \in E$.

Symmetric:

Suppose $\langle (p_1,q_1),(p_2,q_2)\rangle \in E$. Then

$$p_1q_2 = p_2q_1,$$

Since equality is symmetric,

$$p_2q_1 = p_1q_2.$$

So $\langle (p_2,q_2),(p_1,q_1)\rangle \in E$.

Transitive:

Suppose $\langle (p_1,q_1),(p_2,q_2)\rangle \in E$ and $\langle (p_2,q_2),(p_3,q_3)\rangle \in E$.

Then

$$p_1q_2 = p_2q_1, \quad p_2q_3 = p_3q_2.$$

Multiply the first equation by q_3:

$$p_1q_2q_3 = p_2q_1q_3.$$

Multiply the second equation by q_1:

$$p_2q_3q_1 = q_3q_2q_1$$

Thus,

$$p_1q_2q_3 = p_3q_2q_1$$

Since $q_2 > 0$, divide both sides by q_2 we have $p_1q_3 = p_3q_1$.

So $\langle (p_1,q_1),(p_3,q_3)\rangle \in E$.

Therefore, E is an equivalence relation. ∎

12. [2pt] Show that the relation "divides" on $\mathbb{N} \setminus \{0\}$ is a partial order.

A relation is a partial order if it is reflexive, antisymmetric and transitive.

Reflexive:

For any $n > 0$, $n | n$ because $n = n \cdot 1$.

Antisymmetric:

If $a|b$ and $b|a$ then,

$$b = ak, \quad a = bl, \text{ for positive integers } k, l.$$

Substituting gives,

$$a = akl.$$

Thus $kl = 1$.

Since positive integers with product 1 must both be 1, we get $a = b$.

Transitive:

If $a|b$ and $b|c$, then

$$b = ak, \quad c = bl.$$

Thus

$$c = akl,$$

So $a|c$.

Therefore, divides is a partial order.

Every sentence should be complete.

13. [extra 2pt] Find $a, b \in \mathbb{Z}$ such that $31a + 53b = 1$.

Sol'n:

$53 = 31(1) + 22$

$31 = 22(1) + 9$

$22 = 9(2) + 4$

$9 = 4(2) + 1$

Back substitute:

$1 = 9 - 4(2)$

$4 = 22 - 9(2) \Rightarrow 1 = 9 - 2(22 - 2 \cdot 9) = 5 \cdot 9 - 2 \cdot 22$

$9 = 31 - 22(1) \Rightarrow 1 = 5(31 - 22) - 2(22) = 5 \cdot 31 - 7 \cdot 22$

$22 = 53 - 31(1) \Rightarrow 1 = 5 \cdot 31 - 7(53 - 31) = 12 \cdot 31 - 7 \cdot 53$

So one sol'n is

$a = 12, b = -7$

[END]

 Every sentence should be complete.

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	