國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

期末考 June 4, 2025 **Final Exam**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}.$$

(a) [1pt] State the definition of when two matrices are similar.

A and B are similar if $B = Q^{-1}AQ$ for some matrix Q.

(b) [1pt] Is A similar to B? **Provide your reasons.**

No. $A = 2I$ is only similar to itself.

(c) [1pt] Is B similar to C? **Provide your reasons.**

Yes, $C = Q^{-1}BQ$ with $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(d) [1pt] Is B similar to D? **Provide your reasons.**

No, $tr(B) \neq tr(D)$.

(e) [1pt] Find a matrix M such that M is a similar to B and every entry of M is nonzero. **Justify your answer.**

Let $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $Q^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

Then choose $M = Q^{-1}BQ$

$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\cancel{= \begin{bmatrix} 3 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}$

$= \begin{bmatrix} 6 & -4 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$.

2. [5pt] Let

$$A = \begin{bmatrix} 0 & 0 & 30 \\ 1 & 0 & -31 \\ 0 & 1 & 10 \end{bmatrix}.$$

Find a diagonal matrix D and an invertible matrix Q such that $D = Q^{-1}AQ$.

① Find. eigval.

$$\det(A - xI) = \det\begin{bmatrix} -x & 0 & 30 \\ 1 & -x & -31 \\ 0 & 1 & 10-x \end{bmatrix} = -x^2(x-10) + 30 - 31x$$

$$= -x^3 + 10x^2 - 31x + 30$$

$-x^2 + 8x - 15$

$x \quad -2.$

$$= -(x-2)(x-3)(x-5).$$

So $\lambda = 2, 3, 5$.

② Find eigvec.

$\underline{\lambda = 2}$. $A - 2I = \begin{bmatrix} -2 & 0 & 30 \\ 1 & -2 & -31 \\ 0 & 1 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -15 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

$\ker(A-2I) = \operatorname{span}\{\vec{u}_1\}$ with $\vec{u}_1 = \begin{bmatrix} 15 \\ -8 \\ 1 \end{bmatrix}$.

$\lambda = 3$ $A - 3I = \begin{bmatrix} -3 & 0 & 30 \\ 1 & -3 & -31 \\ 0 & 1 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

$\ker(A-3I) = \operatorname{span}\{\vec{u}_2\}$ with $\vec{u}_2 = \begin{bmatrix} 10 \\ -7 \\ 1 \end{bmatrix}$

$\lambda = 5$ $A - 5I = \begin{bmatrix} -5 & 0 & 30 \\ 1 & -5 & -31 \\ 0 & 1 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

$\ker(A-5I) = \operatorname{span}\{\vec{u}_3\}$ with $\vec{u}_3 = \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}$.

So $\underline{D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$, $\underline{Q = \begin{bmatrix} 15 & 10 & 6 \\ -8 & -7 & -5 \\ 1 & 1 & 1 \end{bmatrix}}$

3. [5pt] Solve the recurrence relation

$$a_{n+2} = a_{n+1} + 6a_n,$$
$$a_0 = 0,\ a_1 = 1$$

and find a general form for a_n, in terms of n.

Let $\vec{x}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$. Then $\vec{x}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$

$= A \cdot \vec{x}_n$.

Thus, $\vec{x}_n = A^n \cdot \vec{x}_0$.

Diagonalize A:

Find eigval: $\det(A - xI) = \det\begin{pmatrix} -x & 1 \\ 6 & 1-x \end{pmatrix} = x^2 - x - 6$

$= (x-3)(x+2)$

$\lambda = 3, -2$.

Find eigvec:

$\lambda = 3$. $A - 3I = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \rightsquigarrow u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda = -2$ $A + 2I = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \rightsquigarrow u_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Let $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$. Then $D = Q^{-1}AQ$

and $A = QDQ^{-1}$.

So $\vec{x}_n = QD^nQ^{-1}\vec{x}_0$

$\left(Q^{-1} = -\frac{1}{5}\begin{bmatrix} -2 & -1 \\ -3 & 1 \end{bmatrix} \right)$

$a_n =$ 1-st entry of $QD^nQ^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & \\ & (-2)^n \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underline{\underline{\frac{1}{5} \cdot 3^n - \frac{1}{5}(-2)^n}}$

4. [5pt] Mathematical essay: Write a few paragraphs to introduce the *characteristic polynomial.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. [extra 5pt] Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{bmatrix} \text{ and } J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find an invertible matrix Q such that $J = Q^{-1}AQ$.

Find Q such that $AQ = QJ$.

Let $Q = [u_1 | u_2 | u_3]$. Then

$$AQ = [Au_1 | Au_2 | Au_3].$$

$$QJ = [2u_1 | u_1 + 2u_2 | u_2 + 2u_3].$$

Solve $Au_1 = 2u_1$: $A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

so $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

Solve $Au_2 = u_1 + 2u_2 \Leftrightarrow (A - 2I)\vec{u}_2 = \vec{u}_1$:

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 1 \\ 0 & -2 & 1 & 2 \\ 8 & -12 & 4 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} -2 & 1 & 0 & 1 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

so $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

Solve. $Au_3 = u_2 + 2u_3 \Leftrightarrow (A - 2I)u_3 = \vec{u}_2$

$$\left[\begin{array}{ccc|c|c} -2 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 & \cancel{2} \\ 8 & -12 & 4 & 4 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Chose $Q = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix}$

6. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find an invertibile matrix Q such that $D = Q^{-1}AQ$.

Find basis of $\ker(A-4I)$

$$\Rightarrow u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find basis of $\ker(A-0I)$

$$\Rightarrow u_2, u_3, u_4 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Choose $Q = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	