國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第一次期中考 March 19, 2025 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let A be a 3×3 matrix whose rows are $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Suppose $\det(A) = 3$.

(a) [1pt] Let B be the 3×3 matrix whose rows are $\mathbf{x}, 3\mathbf{x} + \mathbf{y}, \mathbf{z}$. Find $\det(B)$.

$A \xrightarrow{3\rho_1 + \rho_2} B$, so $\det(B) = \det(A) = 3$

(b) [1pt] Let B be the 3×3 matrix whose rows are $\mathbf{x}, \mathbf{y}, 5\mathbf{z}$. Find $\det(B)$.

$A \xrightarrow{5\rho_3} B$, so $\det(B) = 5\det(A) = 15$.

(c) [1pt] Let B be the 3×3 matrix whose rows are $\mathbf{z}, \mathbf{y}, \mathbf{x}$. Find $\det(B)$.

$A \xrightarrow{\rho_1 \leftrightarrow \rho_3} B$, so $\det(B) = (-1)\det(A) = -3$.

(d) [1pt] Let B be the 3×3 matrix whose rows are $\mathbf{y}, \mathbf{z}, \mathbf{x}$. Find $\det(B)$.

$A \xrightarrow{\rho_2 \leftrightarrow \rho_3} \xrightarrow{\rho_1 \leftrightarrow \rho_2} B$, so $\det(B) = (-1)^2 \det(A) = 3$.

(e) [1pt] Let B be the 3×3 matrix whose rows are $\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{x}$. Find $\det(B)$.

✓

$A = \begin{bmatrix} - x - \\ - y - \\ - z - \end{bmatrix} \xrightarrow[\rho_3 + \rho_2]{\rho_2 + \rho_1} \begin{bmatrix} - x+y - \\ - y+z - \\ - z - \end{bmatrix} \xrightarrow{2\rho_3} \begin{bmatrix} - x+y - \\ - y+z - \\ - 2z - \end{bmatrix}$

$\xrightarrow[-\rho_2 + \rho_3]{\rho_1 + \rho_3} \begin{bmatrix} - x+y - \\ - y+z - \\ - z+x - \end{bmatrix} = B$.

, So $\det(B) = 2 \cdot \det(A) = 6$

2. Find the determinant of

$$A = \begin{bmatrix} 1 & -1 & 2 & 2 & -2 \\ -2 & 3 & -6 & -4 & 5 \\ -7 & 9 & -17 & -13 & 14 \\ -15 & 24 & -48 & -30 & 42 \\ 14 & -17 & 32 & 27 & -26 \end{bmatrix}.$$

$$A \xrightarrow[\text{comb's}]{\text{row}} \begin{bmatrix} 1 & -1 & 2 & 2 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & 9 & -18 & 0 & 12 \\ 0 & -3 & 4 & -1 & 2 \end{bmatrix} \xrightarrow[\text{comb's}]{\text{row}} \begin{bmatrix} 1 & -1 & 2 & 2 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow[\text{comb}]{2\rho_3+\rho_5} \begin{bmatrix} 1 & \sim & & & \\ & 1 & \sim & & \\ & & 1 & 1 & -2 \\ & & & 0 & 3 \\ & & & 1 & 1 \end{bmatrix} \xrightarrow{\rho_4 \leftrightarrow \rho_5} \underbrace{\begin{bmatrix} 1 & & & & \\ & 1 & & \sim & \\ & & 1 & & \\ & & & 1 & \\ & & & & 3 \end{bmatrix}}_{= B}$$

$\det(B) = 3$, so $\det(A) = -3$.

3. [2pt] State the definition of the determinant of a matrix, using the row operations.

~~Let~~ The determinant is a function $\det : M_n \to \mathbb{R}$, where M_n is the set of all $n \times n$ real matrices, such that

(1) If $A \xrightarrow{\rho_i \leftrightarrow \rho_j} B$, then $\det(B) = -\det(A)$.

(2) If $A \xrightarrow{k\rho_i} B$, then $\det(B) = k\det(A)$.

(3) If $A \xrightarrow{k\rho_i + \rho_j} B$, then $\det(B) = \det(A)$.

(4) $\det(I_n) = 1$, where I_n is the identity matrix.

4. [1pt] Use the definition to prove that if A is a square matrix whose 5-th row is 10 times its 4-th row, then $\det(A) = 0$.

By assumption, if $A \xrightarrow{-10\rho_4 + \rho_5} B$, then B has its 5-th row zero.

Thus, $\det(B) = 0$ and $\det(A) = 0$ by (3).

5. [2pt] Use the definition to prove that if the rows of A are dependent, then $\det(A) = 0$.

Let $\vec{r}_1, \ldots, \vec{r}_n$ be the rows of A.

~~We~~ Since the rows are dependent, we may assume

$$\vec{r}_j = c_1\vec{r}_1 + c_2\vec{r}_2 + \cdots \cancel{\ } + c_n\vec{r}_n .$$

Thus, $A \xrightarrow[\substack{-c_1\rho_1 + \rho_j \\ -c_2\rho_2 + \rho_j \\ \vdots \\ \cancel{\ } \\ \vdots \\ -c_n\rho_n + \rho_j}]{} B$ gives a zero row to B.

So $\det(B) = 0$ and $\det(A) = 0$ by (3).

6. [5pt] Mathematical essay: Write a few paragraphs to introduce the *permutation expansion.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

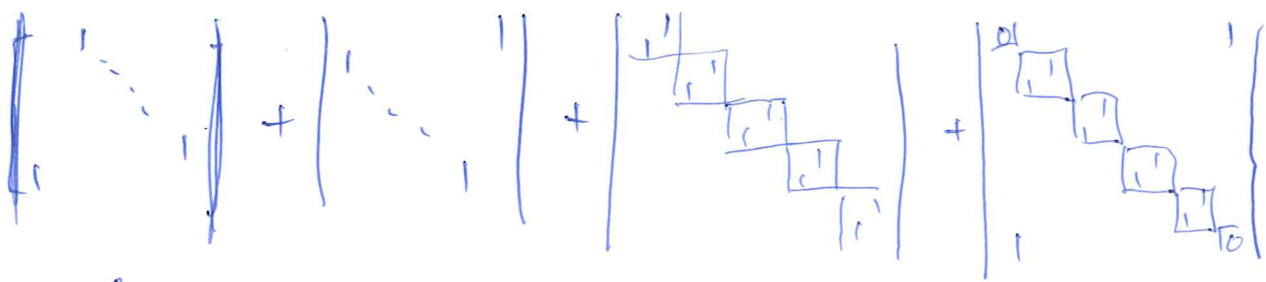
7. [extra 2pt] Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Find $\det(A)$.

By permutation expansion, only the following counts:

$\det(A) =$



$= (-1)^9 + (-1)^9 + (-1)^5 + (-1)^5 = \underline{-4}.$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	