國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） MATH 104A / GEAI 1209A: Linear Algebra II

第二次期中考 April 23, 2025 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ______________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let V be a vector space with a basis $\mathcal{B} = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$. Let W be a vector space with a basis $\mathcal{D} = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$. Suppose $f : V \to W$ is a homomorphism such that

$$\text{Rep}_{\mathcal{B},\mathcal{D}}(f) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}.$$

(a) [1pt] Find $f(\mathbf{x}_1)$.

By $\text{Rep}_{B,D}(f)$, $f(\vec{x}_1) = 1\cdot\vec{y}_1 + 0\vec{y}_2 = \vec{y}_1$

(b) [1pt] Find $f(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$.

$\text{Rep}_B(\vec{x}_1+\vec{x}_2+\vec{x}_3) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so $\text{Rep}_D(f(\vec{x}_1+\vec{x}_2+\vec{x}_3)) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

and $f(\vec{x}_1+\vec{x}_2+\vec{x}_3) = 4\vec{y}_1 + 2\vec{y}_2$.

(c) [1pt] Find a vector $\mathbf{x}$ in V such that $f(\mathbf{x}) = 10\mathbf{y}_1 + 10\mathbf{y}_2$.

Solve $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ and get $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$

So pick $\vec{x} = \vec{x}_1 + 5\vec{x}_2 + 3\vec{x}_3$

(not unique).

(d) [1pt] Find null(f).

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ has 1 free variable, so null$(f) = 1$.

(e) [1pt] Find rank(f).

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ has 2 leading variables, so rank$(f) = 2$.

2. Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most 3 and $\mathcal{B} = \langle 1, x, x^2, x^3 \rangle$ be a basis of $\mathcal{P}_3$. Let $f : \mathcal{P}_3 \to \mathcal{P}_3$ be a homomorphism defined by $p(x) \mapsto p'(x) - xp''(x)$.

(a) [1pt] Find $f(4 + 3x + 2x^2 + x^3)$.

$p'(x) = 3 + 4x + 3x^2$

$x p''(x) = 4x + 6x^2$, so $f(4+3x+2x^2+x^3) = p' - xp''$

$= \underline{3 - 3x^2}$

(b) [2pt] Find the matrix $A = \mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$ such that $A\,\mathrm{Rep}_{\mathcal{B}}(p(x)) = \mathrm{Rep}_{\mathcal{B}}\, f(p(x))$ for all $p(x) \in \mathcal{P}_3$.

$1 \mapsto 0$

$x \mapsto 1$ $\xrightarrow{\text{repr}}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix}$, so $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x^2 \mapsto 0$

$x^3 \mapsto -3x^2$.

(c) [2pt] Find a polynomial $p(x)$ such that $f(p(x)) = 1 + x^2$ and $p(0) = 10$, $p(3) = 4$

Solve $A\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ to get a particular solution $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/3 \end{bmatrix}$

Since $\ker(A) = \left\{ \begin{bmatrix} a \\ 0 \\ c \\ 0 \end{bmatrix} : a, c \in \mathbb{R} \right\}$

General solution $= \left\{ \cancel{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/3 \end{bmatrix}} + \begin{bmatrix} a \\ 1 \\ c \\ -1/3 \end{bmatrix} : a, c \in \mathbb{R} \right\}$.

$p(x) = a + x + cx^2 - \frac{1}{3}x^3$

$p(0) = a = 10$

$p(3) = 10 + 3 + c\cdot 9 - 9 = 4 \Rightarrow c = 0.$

So $p(x) = \underline{10 + x - \frac{1}{3}x^3}$.

3. [2pt] State the definition of an isomorphism.

See textbook.

4. [3pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+2y \\ x+3y \end{bmatrix}.$$

Show that f is an isomorphism.

Claim: f is 1-1.

Suppose $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \neq \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$.

Then $f\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) - f\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+2y_1 \\ x_1+3y_1 \end{bmatrix} - \begin{bmatrix} x_2-2y_2 \\ x_2-3y_2 \end{bmatrix} = \begin{bmatrix} (x_1-x_2)-2(y_1-y_2) \\ (x_1-x_2)-3(y_1-y_2) \end{bmatrix} = \Delta$.

If $\Delta = 0$, then $\begin{cases} (x_1-x_2)-2(y_1-y_2)=0 \\ (x_1-x_2)-3(y_1-y_2)=0 \end{cases}$ conclude that $x_1-x_2=0$ and $y_1-y_2=0$, a contradiction.

Claim: f is onto.

Pick $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Solve $\begin{bmatrix} x+2y \\ x+3y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ to get $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3a-2b \\ b-a \end{bmatrix}$.

Claim: f ~~is~~ preserve structure.

① $f\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+2y_1+2y_2 \\ x_1+x_2+3y_1+3y_2 \end{bmatrix} = \begin{bmatrix} x_1+2y_1 \\ x_1+3y_1 \end{bmatrix} + \begin{bmatrix} x_2+2y_2 \\ x_2+3y_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + f\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$.

② $f\left(r\begin{bmatrix} x \\ y \end{bmatrix}\right) = f\left(\begin{bmatrix} rx \\ ry \end{bmatrix}\right) = \begin{bmatrix} rx+2ry \\ rx+3ry \end{bmatrix} = r\begin{bmatrix} x+2y \\ x+3y \end{bmatrix} = r \cdot f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

5. [5pt] Mathematical essay: Write a few paragraphs to introduce a *homomorphism.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

6. [extra 2pt] Define the plane

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y + 3z \right\}.$$

Find a matrix A such that $A\mathbf{v}$ is the projection of $\mathbf{v}$ onto the plan P for any $\mathbf{v} \in \mathbb{R}^3$.

The projection of v onto $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is

$$\cancel{B} \quad \frac{\langle u, v \rangle}{\|u\|^2} \cdot u = \frac{1}{\|u\|^2} \begin{bmatrix} | \\ u \\ | \end{bmatrix} [-u-] \begin{bmatrix} | \\ v \\ | \end{bmatrix}.$$

So the projection mtx onto u is

$$B = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

Thus, the projection mtx onto P is

$$I - P = \frac{1}{14} \begin{bmatrix} 13 & -2 & -3 \\ -2 & 9 & -6 \\ -3 & -6 & 4 \end{bmatrix}.$$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	