國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第二次期中考 April 23, 2025 **Midterm 2**

姓名 Name : Solution.

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let V be a vector space with a basis $\mathcal{B} = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$. Let W be a vector space with a basis $\mathcal{D} = \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$. Suppose $f : V \to W$ is a homomorphism such that

$$\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f) = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

(a) [1pt] Find $f(\mathbf{x}_1)$.

First column $\Rightarrow f(\vec{x}_1) = \vec{y}_2$.

(b) [1pt] Find $f(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$.

$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, so $f(\vec{x}_1 + \vec{x}_2 + \vec{x}_3) = 2\vec{y}_1 + 4\vec{y}_2$.

(c) [1pt] Find a vector $\mathbf{x}$ in V such that $f(\mathbf{x}) = 10\mathbf{y}_1 + 10\mathbf{y}_2$.

Solve $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ to get $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$.

Pick $\vec{x} = \vec{x}_1 + 5\vec{x}_2 + 3\vec{x}_3$ (not unique)

(d) [1pt] Find null(f).

null(f) = # of free vars = 1

(e) [1pt] Find rank(f).

rank(f) = # of leading vars = 2

2. Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most 3 and $\mathcal{B} = \langle 1, x, x^2, x^3 \rangle$ be a basis of $\mathcal{P}_3$. Let $f : \mathcal{P}_3 \to \mathcal{P}_3$ be a homomorphism defined by $p(x) \mapsto p'(x) - xp''(x)$.

   (a) [1pt] Find $f(2 + 1x + 2x^2 + x^3)$.

   $p'(x) = 1 + 4x + 3x^2$

   $xp''(x) = \quad 4x + 6x^2$, so $f(2+x+2x^2+x^3) = \underline{\underline{1 - 3x^2}}$.

   (b) [2pt] Find the matrix $A = \mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$ such that $A\,\mathrm{Rep}_{\mathcal{B}}(p(x)) = \mathrm{Rep}_{\mathcal{B}}\, f(p(x))$ for all $p(x) \in \mathcal{P}_3$.

   $1 \mapsto 0$

   $x \mapsto 1$

   $x^2 \mapsto 0$

   $x^3 \mapsto -3x^2$

   $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (c) [2pt] Find a polynomial $p(x)$ such that $f(p(x)) = 2 + x^2$ and $p(0) = 5$, $p(3) = 2$

   Solve $\left\{ A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right.$ to get $\underline{\underline{\begin{bmatrix} a \\ 2 \\ c \\ -1/3 \end{bmatrix}}}$

   Choose $p(x) = a + 2x + cx^2 - \frac{1}{3}x^3$

   $p(0) = a = 5$

   $p(3) = 5 + 6 + 9c - 9 = 2 \Rightarrow c = 0$.

   So $\underline{p(x) = 5 + 2x - \frac{1}{3}x^3}$

3. [2pt] State the definition of an isomorphism.

4. [3pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x + 2y \\ x + 3y \end{bmatrix}.$$

Show that f is an isomorphism.

See ver A.

5. [5pt] Mathematical essay: Write a few paragraphs to introduce a *homomorphism.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

6. [extra 2pt] Define the plane

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y + 3z \right\}.$$

Find a matrix A such that $A\mathbf{v}$ is the projection of $\mathbf{v}$ onto the plan P for any $\mathbf{v} \in \mathbb{R}^3$.

See ver A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	