MINIMUM RANK OF POWERS OF CYCLES AND TREES

GERARD JENNHWA CHANG[†] AND JEPHIAN CHIN-HUNG LIN[‡]

Abstract. The minimum rank $\mathrm{mr}(G)$ (respectively, maximum nullity $M(G)$) of a graph G with n vertices is the minimum rank (respectively, maximum nullity) of an $n \times n$ real symmetric matrix A with its off-diagonal entry $A_{ij} \neq 0$ whenever ij is an edge of G. L. de Alba et. al. [2] studied the minimum rank problem on the powers of paths and the strict powers of trees. This paper continues the research and gives an explicit formula for the minimum rank of powers of cycles and powers of trees.

Key words. Minimum Rank, Maximum Nullity, Zero forcing number, Path cover number, Power, Cycle, Tree.

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1. Introduction. Graphs and symmetric matrices are in intimate relation. For an $n \times n$ real symmetric matrix A, it is natural to consider the corresponding graph $G = \mathcal{G}(A)$ with

$$\text{vertex set } V(G) = \{1, 2, \dots, n\} \text{ and edge set } E(G) = \{ij : i \neq j, A_{ij} \neq 0\},$$

where A_{ij} is the ij-entry of A. Conversely, for a graph G with n vertices, there is a class of $n \times n$ real symmetric matrices whose corresponding graph is G. Denote this class as

$$\mathcal{S}(G) = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A = A^\top,\ \mathcal{G}(A) = G\},$$

where $\mathcal{M}_{n\times n}(\mathbb{R})$ is the set of all $n \times n$ matrices over the field of real numbers. The *minimum rank* of a graph G is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}(G)\};$$

and the *maximum nullity* of G is

$$M(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}(G)\}.$$

It is easy to see that

$$\mathrm{mr}(G) + M(G) = |V(G)|.$$

So a result in $\mathrm{mr}(G)$ can be presented as a result in $M(G)$ and vice versa. In this paper we very often write results in terms of $M(G)$ rather than $\mathrm{mr}(G)$.

For a positive integer r, the *r-th power* of a graph G is the graph G^r whose vertex set is $V(G)$ and two distinct vertices i and j are adjacent in G^r if their distance in G is at most r. The maximum nullity of the path P_n of n vertices is 1. de Alba et al. [2] proved that $M(P_n^r) = \min\{r, n-1\}$. It is also known that the maximum nullity of cycle C_n of n vertices is 2 for $n \geq 3$. Nazari and Radpoor [7] proved that $M(C_n^r) = 2r$ for $r \leq \frac{n}{2}$ by using the *delta Conjecture* that $\delta(G) \leq M(G)$ for any graph G, which was posted in [4] but remains unsolved. In Section 3, we prove this result without using the delta Conjecture. In Section 4, we determine the maximum nullity of the square of a tree.

[†]Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan (gjchang@math.ntu.edu.tw) and National Center for Theoretical Sciences, Taipei Office, Taiwan. This research is partially supported by the National Science Council of the Republic of China under grant NSC101-2115-M-002-005-MY3.

[§]National Center for Theoretical Sciences, Taipei Office, Taiwan (jlch3554@hotmail.com).

2. Notation and terminology. For a positive integer n, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. The *support* $\operatorname{supp}(v)$ of a vector $v \in \mathbb{R}^n$ is the index set of nonzero entries of v.

A *zero forcing set* of a graph G is a subset $F \subseteq V(G)$ which can force all vertices black at the end of repeatedly applying the following *color changing rule*:

- initially, all vertices in F are black and all other vertices are white;
- if a black vertex x has exactly one white neighbor y, then y is changed to be black.

The *zero forcing number* $Z(G)$ is the minimum size of a zero forcing set of G. In the above rule, we write $x \to y$ to refer that a black vertex x forces its only white neighbor y to be black. A *chronological list* is a chronological record $\{x_i \to y_i\}_{i=1}^{s}$, where $x_i \to y_i$ is the color changing at iteration i. A *zero forcing process* ζ refers to a zero forcing set together with the corresponding chronological list. For more detail on the parameter Z, see [3]. The following inequality from [1] is particularly useful in this paper: for any graph G,

$$M(G) \le Z(G). \tag{2.1}$$

A *path cover* of a graph G is a collection $\mathcal{P}$ of disjoint induced paths that cover all vertices of G. The *path cover number* $p(G)$ of G is the minimum size of a path cover of G. It is known that $M(G) = p(T)$ for a tree T [6] and $M(G) \le p(G)$ for an outerplanar graph G [8]. For a positive integer r, the *r-th weight* of a path cover $\mathcal{P}$ is

$$w_r(\mathcal{P}) = \sum_{\pi \in \mathcal{P}} Z(\pi^r);$$

and the *r-th path cover number* of G is

$$p_r(G) = \min\{w_r(\mathcal{P}) : \mathcal{P} \text{ is a path cover of } G\}.$$

Since $Z(\pi) = 1$ for any path π, it is the case that $p_1(G) = p(G)$.

The *clique cover number* $\operatorname{cc}(G)$ of a graph G is the minimum number of (not necessarily disjoint) cliques to cover $E(G)$. It is known that $\operatorname{cc}(G) \ge \operatorname{mr}(G)$ for all G, even we replace the field $\mathbb{R}$ by any other infinite field, see [5]. The *star-clique cover* $\mathcal{C}$ of a graph G is a set of stars and cliques that cover all edges of G. The *weight* of $\mathcal{C}$ is $w(\mathcal{C}) = 2p + q$ when $\mathcal{C}$ consists of p stars and q cliques. The *star-clique number* of G is

$$\operatorname{scc}(G) = \min\{w(\mathcal{C}) : \mathcal{C} \text{ is a star-clique cover of } G\}.$$

By the facts that $\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$ and that $\operatorname{mr}(K_{a,b}) = 2$ for $a + b \ge 3$, it follows that $\operatorname{scc}(G) \ge \operatorname{mr}(G)$ for any graph G. The *dual star-clique cover number* is defined as $\overline{\operatorname{scc}}(G) = |V(G)| - \operatorname{scc}(G)$. Then, for any graph G,

$$\overline{\operatorname{scc}}(G) \le M(G) \le Z(G). \tag{2.2}$$

Section 4 shows that $\overline{\operatorname{scc}}(T^2) = M(T^2) = Z(T^2) = p_2(T)$ for any tree T.

3. Powers of cycles. Recall that Nazari and Radpoor [7] proved that $M(C_n^r) = 2r$ for $r \le \frac{n}{2}$ by using the delta Conjecture that $\delta(G) \le M(G)$ for any graph G, which was posted in [4] but remains unsolved. The purpose of this section is to give a proof of this result without using the delta Conjecture. The following lemma in [4] is useful.

Lemma 3.1. *For positive integers $k \le n$, there is a $k \times n$ real matrix C whose $k \times k$ submatrices are nonsingular. Also, S is the support of a non-zero vector v with $Cv = 0$ if and only if $|S| \ge k + 1$.*

Theorem 3.2. *If $n \ge 3$, then $M(C_n^r) = Z(C_n^r) = \min\{2r, n-1\}$.*

Proof. If $2r \ge n-1$, then $C_n^r = K_n$ and so $M(C_n^r) = Z(C_n^r) = n-1$.

We now consider the case of $2r \le n-2$. Since each set of $2r$ consecutive vertices of the cycle form a zero forcing set, $M(C_n^r) \le Z(C_n^r) \le 2r$. Next, we shall prove that $M(C_n^r) \ge 2r$ by constructing a symmetric matrix A with $\mathcal{G}(A) = C_n^r$ and $\operatorname{rank}(A) \le n-2r$.

For $k \in [n-r]$, let I_k be the subset $\{k, k+1, \dots, k+r\}$ of $[n-r]$, where the addition is taken module $n-r$, that is, $k+i$ is $k+i-(n-r)$ if $k+i > n-r$. By Lemma 3.1, we may choose an $r \times (n-r)$ real matrix C whose $r \times r$ submatrices are nonsingular; also a vector $v_k \in \mathbb{R}^{n-r}$ such that $Cv_k = 0$ with $\operatorname{supp}(v_k) = I_k$ for each $k \in [n-r]$. Next, choose appropriate coefficients a_k such that $A = \sum_{i=1}^{n-2r} a_i v_i v_i^\top$ has the property that $\mathcal{G}(A) = P_{n-r}^r$. This is possible because we only have to worry about that some nonzero entries vanish under the process of summation. However, there are only finitely many these conditions and we have infinitely many choices for the coefficients. Furthermore, we can choose a_{n-2r} as small as we want.

Let B be the $(n-r) \times r$ matrix whose i-th column is v_{n-2r+i}. Since all v_i are in the null space of C, the space spanned by $\{v_1, v_2, \dots, v_{n-r}\}$ has dimension at most $n-2r$. Also, $\operatorname{rank}(A) \ge n-2r$, since $\operatorname{mr}(P_{n-r}^r) = n-2r$. Hence, there is a matrix X such that $AX = B$. As we may choose a_{n-2r} as small as we want, $X^\top AX$ can be chosen to contain no zero entries. Then

$$D = \begin{bmatrix} A & B \\ B^\top & X^\top AX \end{bmatrix} = \begin{bmatrix} A & B \\ X^\top A & X^\top B \end{bmatrix}$$

has rank $n-2r$ and $\mathcal{G}(D) = C_n^r$. These prove that $M(C_n^r) \ge 2r$ and so $M(C_n^r) = Z(C_n^r) = 2r$ as desired. □

4. Squares of trees. The purpose of this section is to determine the maximum nullity of the square of a tree. Besides a formula in terms of the zero forcing number, a procedure to compute it is also given.

THEOREM 4.1. *If T is a tree, then $\overline{\operatorname{scc}}(T^2) = M(T^2) = Z(T^2) = p_2(T)$.*

The theorem follows from (2.2) that $\overline{\operatorname{scc}}(T^2) \le M(T^2) \le Z(T^2)$ and the following two lemmas. The first lemma proves that $p_2(T)$ is an upper bound of $Z(T^2)$ and the second lemma proves that $p_2(T)$ is a lower bound of $\overline{\operatorname{scc}}(T^2)$.

LEMMA 4.2. *If T is a tree, then $Z(T^2) \le p_2(T)$.*

Proof. Choose a path cover $\mathcal{P}$ of T with $w_2(\mathcal{P}) = p_2(T)$. A vertex is *special* if it is of degree 2 and is adjacent to a leaf. We conclude the lemma by proving Claim 1 using an induction on the number n of vertices of T.

Claim 1. *$|F| \le w_2(\mathcal{P})$ for some zero forcing set F of T^2 with a zero forcing process ζ for which each forcing $x \to y$ has the property that $d_T(x,y) = 2$ whenever y is not special in T.*

The claim is clear for $n = 1$. Now assume that $n \ge 2$. For the case when T is a star, T^2 is K_n. Choose F as a set of all vertices except a leaf of T. The claim follows from that $\mathcal{P}$ consists of paths with minimum total weight $n-1$. For the case when T is the n-path $v_1, v_2, \dots, v_n$ with $n \ge 3$, choose $F = \{v_1, v_2\}$ which is a zero forcing set of T^2 using the chronological list $\{v_i \to v_{i+2}\}_{i=1}^{n-2}$. Then $Z(T^2) \le 2$. As $Z(T^2) \ge \delta(T^2) = 2$. In fact $Z(T^2) = 2$. The claim then follows.

Now we consider the case when T is neither a star nor a path. In this case, there is always a path $\pi\colon v_1, v_2, \dots, v_r, \dots, v_s$ in $\mathcal{P}$ such that v_1 is a leaf in T and v_r is the only vertex of π adjacent to an unique vertex $v \notin \pi$ in T. Then, $T_1 \coloneqq T - \pi$ is a tree of at least 3 vertices and $\mathcal{P}_1 \coloneqq \mathcal{P} - \{\pi\}$ is a path cover of T_1. We claim that v is not a special vertex in T_1 when $s = 2$. Suppose to the contrary that $s = 2$ but v is of degree 2 and is adjacent to a leaf u in T_1. Suppose u is a vertex of a path $\pi_1 \in \mathcal{P}$.

Case 1. $\pi_1 = u$. Suppose v is in path $\pi_2 \in \mathcal{P}$. In this case, we may replace $\mathcal{P}$ by the path cover $(\mathcal{P} - \{\pi_2, \pi\}) \cup \{\pi_2 + \pi\}$ of weight no more than $\mathcal{P}$ and replace π by u.

Case 2. $\pi_1 = uv$. In this case, we may replace $\mathcal{P}$ by the path cover $(\mathcal{P} - \{\pi_1, \pi\}) \cup \{\pi_1 + \pi\}$ of weight no more than $\mathcal{P}$ and replace π by $\pi_1 + \pi$.

Case 3. π_1 has at leas 3 vertices. In this case, we may replace $\mathcal{P}$ by the path cover $(\mathcal{P} - \{\pi_1, \pi\}) \cup \{u, (\pi_1 - u) + \pi\}$ of weight no more than $\mathcal{P}$ and replace π by u. So, we may assume that either $s \neq 2$ or v is not special. By the induction hypothesis, $|F + 1| \leq w_2(\mathcal{P}_1)$ for some zero forcing set F_1 of T_1^2 with a zero forcing process ζ_1 for which each forcing $x \to y$ has the property that $d_{T_1}(x, y) = 2$ whenever y is not special in T_1, in particular when y is v for the case of $s = 2$.

Let $F = F_1 \cup \{v_1\}$ when $s \leq 2$ and let $F = F_1 \cup \{v_1, v_2\}$ when $s \geq 3$. We shall check that F is a zero forcing set of T^2 by constructing a zero forcing process corresponding to F as follows. First, if $s \geq 3$, then do forcing $v_i \to v_{i+2}$ for $1 \leq i \leq r - 2$. By now, v_r is black unless $s = 2$. Next, do all forcing $x \to y$ of ζ_1 until $v^* \to v$. Notice that either v_r is black or else $s = 2$ and so $d_{T_1}(v^*, v) = 2$. In either case, all the forcing of ζ_1 mentioned above do not infect the vertices in π. Then do the forcing $v_1 \to v_2$ when $s = 2$ or the forcing $v_{r-1} \to v_{r+1}$ when $s \geq 3$. Finally, do the remaining forcing of ζ_1, follow by the remaining forcing $v_i \to v_{i+2}$ alone π for $r \leq i \leq s - 2$. These give a zero forcing processing corresponding to F with the property that $d_T(x, y) = 2$ whenever y is not special in T. This completes the proof of Claim 1. $\square$

For a vertex v in T, we use $N(v)$ and $N^2(v)$ to denote the set of neighbors of v in T and T^2 respectively. We use κ_v to denote the clique induced by $N(v) \cup \{v\}$ in T^2, and use σ_v to denote the star in T^2 whose center is v and whose set of leaves is $N^2(v)$.

In a tree T, a *pendent path* is a maximal induced path that contains a leaf but no vertex of degree more than 2. A *pendent branch* consists of vertex v with degree $k + 1 \geq 3$ and k pendent paths each has an end vertex adjacent to v. For the case when T is not a path, a pendent branch can be obtained from a breadth first search. Equivalently, consider T rooted at a chosen vertex r and choose a vertex v of degree $k + 1 \geq 3$ farest from r. Then v has k children and all proper descendants of v form k pendent paths of the pendent branch.

LEMMA 4.3. *If T is a tree, then $p_2(T) \leq \overline{\mathrm{scc}}(T^2)$.*

Proof. We shall prove the lemma by induction on the number n of vertices of T. For the case when T is a path $\pi: v_1, v_2, \ldots, v_n$, the lemma follows from considering the path cover $\{\pi\}$ and the star-clique cover $\{\kappa_{v_i}: 2 \leq i \leq \max\{2, n-1\}\}$. Now assume that T is not a path and so $n \geq 4$. Choose a pendent branch at a vertex v of degree $k + 1 \geq 3$ with pendent paths $\alpha_i: v_1^i, v_2^i, \ldots, v_{s_i}^i$ for $1 \leq i \leq k$. We consider three cases.

Case 1. One of the following conditions holds: (i) there is some $s_i \geq 4$, (ii) there is some $s_i = 2$, (iii) $k = 2$ and there is some $s_i = 3$ with the other $s_j = 1$. Let T_1 be the tree obtained from T by deleting the leaf $v_{s_i}^i$. By the induction hypothesis, $p_2(T_1) \leq \overline{\mathrm{scc}}(T_1^2)$. Choose a path cover $\mathcal{P}_1$ of T_1 with $w_2(\mathcal{P}_1) = p_2(T_1)$ and a star-clique cover $\mathcal{C}_1$ of T_1^2 with $w(\mathcal{C}_1) = \mathrm{scc}(T_1^2)$. Let π be the path in $\mathcal{P}_1$ which contains the leaf $v_{s_i-1}^i$ in T_1. If $|V(\pi)| = 2$, then we change $\mathcal{P}_1$ and π according to three subcases:

(i) The other end vertex $v_{s_i-2}^i$ of π is adjacent to $v_{s_i-3}^i$ which is an end vertex of a path π_1 in $\mathcal{P}_1$. Since $Z((\pi_1 + \pi)^2) \leq 2 \leq Z(\pi_1^2) + Z(\pi^2)$, we may replace $\mathcal{P}_1$ by $(\mathcal{P}_1 - \{\pi_1, \pi\}) \cup \{\pi_1 + \pi\}$ and replace π by $\pi_1 + \pi$.

(ii) The other end vertex v of π is adjacent to a neighbor of v which is an end vertex of a path π_1 in $\mathcal{P}_1$. Replace $\mathcal{P}_1$ by $(\mathcal{P}_1 - \{\pi_1, \pi\}) \cup \{\pi_1 + \pi\}$ and replace π by $\pi_1 + \pi$.

(iii) The other end vertex v_1^i of π is adjacent to v. Let v is in a path π_1 in $\mathcal{P}_1$. If v is an end vertex of π_1, then replace $\mathcal{P}_1$ by $(\mathcal{P}_1 - \{\pi_1, \pi\}) \cup \{\pi_1 + \pi\}$ and replace π by $\pi_1 + \pi$. If v is not an end vertex of π_1, then the leaf v_1^j is an end vertex of π_1. In this case, replace $\mathcal{P}_1$ by $(\mathcal{P}_1 - \{\pi_1, \pi\}) \cup \{v_1^j, (\pi_1 - v_1^j) + \pi\}$ and replace π by $(\pi_1 - v_1^j) + \pi$.

By now we may assume that $|V(\pi)| \neq 2$. Then $\mathcal{P} := (\mathcal{P}_1 - \{\pi\}) \cup \{\pi + v_{s_i}^i\}$ is a path cover of T with $w_2(\mathcal{P}) = w_2(\mathcal{P}_1)$, since $Z((\pi + v_{s_i}^i)^2) = Z(\pi^2) = 1$ if $|V(\pi)| = 1$ and $Z((\pi + v_{s_i}^i)^2) = Z(\pi^2) = 2$ if $|V(\pi)| \geq 3$. Also $\mathcal{C} := \mathcal{C}_1 \cup \{\kappa_{v_{s_i-1}^i}\}$ is a star-clique cover of $\mathcal{C}$ with $w(\mathcal{C}_1) + 1 = w(\mathcal{C})$. Consequently,

$$p_2(T) \leq w_2(\mathcal{P}) = w_2(\mathcal{P}_1) \leq \overline{\mathrm{scc}}(T_1^2) = |V(T_1)| - w(\mathcal{C}_1) = |V(T)| - w(\mathcal{C}) \leq \overline{\mathrm{scc}}(T^2).$$

Case 2. $k = 2$ and $s_1 = s_2 = 3$. Let T_1 be the tree obtained from T by deleting α_1, v and α_2. By the induction hypothesis, $p_2(T_1) \leq \overline{\mathrm{scc}}(T_1^2)$. Choose a path cover $\mathcal{P}_1$ of T_1 with $w_2(\mathcal{P}_1) = p_2(T_1)$ and a star-clique cover $\mathcal{C}_1$ of T_1^2 with $w(\mathcal{C}_1) = \mathrm{scc}(T_1^2)$. Then $\mathcal{P} := \mathcal{P}_1 \cup \{\alpha_1 + v + \alpha_2\}$ is a path cover of T with $w_2(\mathcal{P}) = w_2(\mathcal{P}_1) + 2$. Also $\mathcal{C} := \mathcal{C}_1 \cup \{\sigma_v, \kappa_v, \kappa_{v_2^1}, \kappa_{v_2^2}\}$ is a star-clique cover of T^2 with $w(\mathcal{C}_1) + 5 = w(\mathcal{C})$. Consequently,

$$p_2(T) \leq w_2(\mathcal{P}) = w_2(\mathcal{P}_1) + 2 \leq \overline{\mathrm{scc}}(T_1^2) + 2 = |V(T_1)| - w(\mathcal{C}_1) + 2 = |V(T)| - w(\mathcal{C}) \leq \overline{\mathrm{scc}}(T^2).$$

Case 3. One of the following conditions holds: (i) $k = 2$ with $s_1 = s_2 = 1$, (ii) $k \geq 3$ with some s_i say $s_1 = 1$, (iii) $k \geq 3$ with all $s_i = 3$. Let T_1 be the tree obtained from T by deleting α_1. By the induction hypothesis, $p_2(T_1) \leq \overline{\mathrm{scc}}(T_1^2)$. Choose a path cover $\mathcal{P}_1$ of T_1 with $w_2(\mathcal{P}_1) = p_2(T_1)$ and a star-clique cover $\mathcal{C}_1$ of T_1^2 with $w(\mathcal{C}_1) = \mathrm{scc}(T_1^2)$. We have the following facts.

(a) We may assume that if $\sigma_x \in \mathcal{C}_1$, then x has degree at leat 3 in T_1. For otherwise we may replace σ_x by κ_y for all $y \in N_{T_1}(x)$ to get a star-clique cover of weight no more than $\mathcal{C}_1$.

(b) We may assume that if x has at least two neighbors y_1 and y_2 of degree 1 or 2 in T_1, then $\kappa_x \in \mathcal{C}_1$. As the edge $y_1 y_2$ can be covered only by κ_x, σ_{y_1} or σ_{y_2}, this follows from (a).

(c) Under condition (i), we may assume that $\kappa_v \in \mathcal{C}_1$. This follows from the facts that the edge vv_1^2 can only be covered by κ_v or $\kappa_{v_1^2}$ and that κ_v covers more edges than $\kappa_{v_1^2}$.

(d) Under condition (iii), we may assume that $\sigma_v \in \mathcal{C}_1$. For otherwise $\kappa_{v_1^i}$ and $\kappa_{v_2^i}$ are in $\mathcal{C}_1$ for $2 \leq i \leq k$ and so we may replace $\kappa_{v_1^i}$ for $2 \leq i \leq k$ by σ_v to get a star-clique cover of weight no more than $\mathcal{C}_1$.

Now $\mathcal{P} := \mathcal{P}_1 \cup \{\alpha_1\}$ is a path cover of T with $w_2(\mathcal{P}) = w_2(\mathcal{P}_1) + Z(\alpha_1^2)$. According to (a), (b), and (c), $\kappa_v \in \mathcal{C}_1$ in any case. According to (d), $\sigma_v \in \mathcal{C}_1$ under condition (iii). Under condition (i) or (ii), $\mathcal{C} := (\mathcal{C}_1 - \{\kappa_v \text{ in } T_1\}) \cup \{\kappa_v \text{ in } T\})$ is a clique cover of T with $w(\mathcal{C}_1) = w(\mathcal{C})$. Under condition (iii), $\mathcal{C} := (\mathcal{C}_1 - \{\kappa_v \text{ and } \sigma_v \text{ in } T_1\}) \cup \{\kappa_v$ and σ_v in $T, \kappa_{v_2^1}\})$ is a clique cover of T with $w(\mathcal{C}_1) + 1 = w(\mathcal{C})$. In any case, $w(\mathcal{C}_1) + |V(\alpha_1)| - Z(\alpha_1^2) = w(\mathcal{C})$. Hence,

$$p_2(T) \leq w_2(\mathcal{P}) = w_2(\mathcal{P}_1) + Z(\alpha_1^2) = p_2(T_1) + Z(\alpha_1^2) \leq \overline{\mathrm{scc}}(T_1^2) + Z(\alpha_1^2)$$

$$= |V(T_1)| - w(\mathcal{C}_1) + Z(\alpha_1^2) = |V(T)| - w(\mathcal{C}) = \overline{\mathrm{scc}}(T^2).$$

This completes the proof. $\square$

The proof of the theorem in fact provides an algorithm for computing $M(T^2)$. We summary it as follows.

Corollary 4.4. *If T is a path, then $M(T^2) = 1$ when $|V(T)| \leq 2$ and $M(T) = 2$ when $|V(T)| \geq 3$. If T is a tree containing a pendent branch $\mathcal{B}$ which has p pendent paths with at most 2 vertices and q paths of at least 3 vertices, T_1 is obtained from T by deleting $\mathcal{B}$ and T_2 is obtained from T by replacing $\mathcal{B}$ with a path of two vertices, then $M(T) = M(T_1) + p + 2q - 2$ if $q \geq 2$ and $M(T) = M(T_2) + p + q - 1$ if $q \leq 1$.*

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