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零強迫數在最小秩問題上的應用

Applications of zero forcing number to the minimum rank problem

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誌謝

學習，很多時候都是自己的事，但從沒人說過，在有人陪伴時不會表現得更好。我很慶幸在兩年的碩士班生活中，有許多人的一路陪伴。很感謝指導教授張鎮華老師收我當學生，讓我學到許多做學問的方法與做人處事的態度；也很感謝老師的關懷，讓我保持對學習與研究的熱情。

做研究的過程中，和學長姊一起參加研討會讓我學到了許多東西。硯仁和聖華學長總是不厭其煩地陪我討論問題，往往在討論過程中會有意想不到的想法，也讓自己覺得不是一個人在奮鬥。另外中央大學的葉鴻國教授與黃良豪學長，常常在論文相關問題上給許多回饋。口試委員顏經和教授也對論文提出一些建議。這些都是不可或缺的要素，所以非常謝謝你們。

最後，最重要的還是謝謝家人們一直以來的支持與鼓勵，以及女朋友的陪伴。

中文摘要

一個圖G的最小秩問題是在討論所有可決定G的實對稱方陣中最小的秩，這等同於討論這群方陣中的最大零維數$M(G)$。零強迫數$Z(G)$是指最小零強迫集的個數，可用於最小秩問題的研究。而路徑覆蓋數$P(G)$是指可以用來覆蓋圖G點集的最小導出路徑數。當圖G中有截點時，我們提出一個公式用小圖的零強迫數來計算原圖G的零強迫數，並且討論在某些條件下$P(G)$會等於$Z(G)$，而這條件叫做強PZ條件。

零強迫數$Z(G)$是$M(G)$已知的上界。我們提出一個更緊的上界叫窮舉零強迫數$\widetilde{Z}(G)$，也就是$Z(G) \geq \widetilde{Z}(G) \geq M(G)$。並且提出一個篩選過程，使得在某些特殊例子中，可以得到比窮舉零強迫數再更緊的上界。

最後，我們找到一個反例，可以用來回答一個關於零強迫數在邊上的差值問題。

Abstract

The minimum rank problem of a graph G is to determine the smallest rank over all real symmetric matrices whose ij-entry, $i \neq j$, is nonzero whenever ij is an edge and is zero otherwise. Equivalently, it is the same to determine the largest nullity over those matrices. This value is called the maximum nullity $M(G)$. The zero forcing number $Z(G)$ is the minimum size of a zero forcing set and is useful on studying the minimum rank problem. And the path cover number $P(G)$ is the minimum number of vertex disjoint induced paths which cover the vertices of G. In the case that G is a graph with cut-vertices, we give a reduction formula to compute the zero forcing number $Z(G)$ by the zero forcing numbers of its subgraphs. We also discuss a condition called the strong PZ condition such that $P(G)$ is equal to $Z(G)$ under this condition.

It is well known that $Z(G) \geq M(G)$ for all graphs G. In this thesis, we introduce a sharper upper bound, the exhaustive zero forcing number $\widetilde{Z}(G)$, which satisfies $Z(G) \geq \widetilde{Z}(G) \geq M(G)$ for all graphs G. Furthermore, by doing a operation called the sieving process, we get some upper bounds less than $\widetilde{Z}(G)$ for some special cases.

Finally, we give an example to answer a question about the edge spread of the zero forcing number.

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1 Introduction

As the concept of incidence matrix showing, there is a natural relation between symmetric matrices and graphs. For an $n \times n$ real symmetric matrix A, we consider the corresponding graph $G = \mathcal{G}(A)$ with

$$\text{vertex set } V(G) = \{1, 2, \dots, n\} \text{ and edge set } E(G) = \{ij : A_{ij} \neq 0, i \neq j\},$$

where A_{ij} is the ij-entry of A. On the other hand, for any given graph G, there is a class of real symmetric matrices whose corresponding graph is G. Denote this class as

$$\mathcal{S}(G) = \{A \in M_{n\times n}(\mathbb{R}) : A = A^\top, \mathcal{G}(A) = G\},$$

where $M_{n\times n}(\mathbb{R})$ is the set of $n \times n$ matrices over the field of real numbers. The *minimum rank problem* is to determine the minimum rank over all matrices in the set $\mathcal{S}(G)$ for a given graph G. This value is called the *minimum rank* of the graph G and is denoted by

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}(G)\}.$$

The minimum rank problem comes from a more general problem called the *inverse eigenvalue problem*. It relates to many topics in other fields. For more detail, the motivations and the applications, please see the surveys [9, 10].

Minimum ranks of some graphs are well known, see a table in paper [1]. For example, the minimum rank of an n-path is $\mathrm{mr}(P_n) = n - 1$ for $n \geq 1$ and of an n-cycle is $\mathrm{mr}(C_n) = n - 2$ for $n \geq 3$. Also, the minimum rank of the complete graph K_n is $\mathrm{mr}(K_n) = 1$ and of the complete bipartite graph $K_{m,n}$ is $\mathrm{mr}(K_{m,n}) = 2$ for $m + n \geq 2$ while $\mathrm{mr}(K_{1,1}) = 1$.

To study the minimum rank of a graph, several related parameters are considered. The *maximum nullity*, denoted by $M(G)$, of a graph G is defined by

$$M(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}(G)\}.$$

It is easy to see that

$$\mathrm{mr}(G) + M(G) = |V(G)|.$$

So in this article we very often write results in term of $M(G)$ rather than $\mathrm{mr}(G)$. A quite powerful concept called the zero forcing number is defined below.

Definition 1. The *zero forcing process* on a graph G is the color-changing process using the following rules.

- Each vertex of G is either black or white initially.
- If x is black and y is the only white neighbor of x, then change the color of y to black.

A set $F \subseteq V(G)$ is called a *zero forcing set* if with the initial condition that the vertices in F are black and the remaining vertices are white, each vertex of G could be forced into black by the zero forcing process. The *zero forcing number* $Z(G)$ of a graph G is the minimum size of a zero forcing set.

It was shown in [1] that $M(G) \leq Z(G)$ for all graphs G. In fact, most of graphs whose maximum nullity are known have the fact $M(G) = Z(G)$. This means that $Z(G)$ is a sharp upper bound of $M(G)$ in some sense. But this also means that for those graphs such that $Z(G)$ is not a sharp upper bound of $M(G)$, we are less familiar with them.

Another related parameter is the *path cover number* $P(G)$ of a graph G, which is the minimum number of vertex disjoint induced paths of G that cover all vertices of G. It was shown in [2] that $P(G) \leq Z(G)$ for all graphs G. Although in general $M(G)$ and $P(G)$ are not comparable, it was shown in [14] that $M(G) \leq P(G)$ if G is outerplanar. So $P(G)$ is a better upper bound than $Z(G)$ for $M(G)$ when G is outerplanar.

In the case when G contains cut-vertices, it is much easier to determine the above parameters for G by determining those for blocks of G. We give a vertex reduction formula for the zero forcing number in Section 2. We establish a property which is preserved by vertex-sum operation and hereditary for induced subgraphs in Section 3. We then give an example for which the difference $Z(G) - M(G)$ could be arbitrarily large in Section 4.

There is another aspect to consider $\mathcal{S}(G)$ as a set of matrices whose entries are confined to be zero, nonzero or unknown. We study the minimum rank of the set of matrices whose entries are confined in Sections 5 and 6. With this aspect, we derive a new upper bound of $M(G)$ called the exhaustive zero forcing number in Sections 7 and 8.

Finally, we give a summary about upper bounds on $M(G)$ in Section 9. We also give an example to answer a question proposed in [8].

2 Vertex reduction for $Z(G)$ and $P(G)$

For two graphs G_1 and G_2 with a vertex labeled v on each of them, the graph G obtained from $G_1 \cup G_2$ by identifying the vertices labeled v is called the *vertex-sum* of G_1 and G_2 at v. We write it as $G = G_1 \oplus_v G_2$. There is a formula to compute the maximum nullity of the vertex-sum by that of the original graphs.[1]

Theorem 1 ([4]). *If $G = G_1 \oplus_v G_2$, then the maximum nullity of G is*

$$M(G) = \max\{M(G_1) + M(G_2) - 1, M(G_1 - v) + M(G_2 - v) - 1\}.$$

There is an alternative expression for the formula in Theorem 1 in terms of null spread defined below.

Definition 2. For any vertex v in a graph G, the *null spread* $m_v(G)$, the *path spread* $p_v(G)$, and the *zero spread* $z_v(G)$ of G at v are defined by:

$$m_v(G) = M(G) - M(G - v),$$

$$p_v(G) = P(G) - P(G - v),$$

$$z_v(G) = Z(G) - Z(G - v).$$

Having this definition in mind, the formula in Theorem 1 then can be rewritten as

$$m_v(G) = \max\{m_v(G_1) + m_v(G_2), 0\} - 1$$

by noting that $M(G - v) = M(G_1 - v) + M(G_2 - v)$. For the formula on path cover numbers, we need the following definition.

Definition 3. A vertex v is *doubly P-terminal* if there is an optimal path cover such that v is the only vertex in the path who covers v. A vertex v is *simply P-terminal* if there is an optimal path cover such that v is an endpoint of the path who covers v and v is not doubly P-terminal.

We remark that the terminologies "doubly P-terminal" and "simply P-terminal" were originally called "doubly terminal" and "simply terminal" in [5]. The purpose of using P-terminal is just to distinguish the concept Z-terminal defined later.

Then, the formula for path spread is as follows.

[1]For the result on the sum at two vertices, see [12].

Theorem 2 ([5]). *If $G = G_1 \oplus_v G_2$, then the path spread of G at v is*

$$p_v(G) = \begin{cases} -1, & \text{if } v \text{ is simply P-terminal for } G_1 \text{ and } G_2; \\ \min\{p_v(G_1), p_v(G_2)\}, & \text{otherwise.} \end{cases}$$

Now we use the similar language to deduce the formula for the zero forcing number, since the zero forcing number has some properties related to induced paths.

Definition 4. A *chronological list* of a zero forcing set is the order of forces given by the zero forcing process. A *chain* of a chronological list is a sequence of vertex such that

$$v_1 \to v_2 \to \cdots \to v_k,$$

where the arrows means the former changes the color of the latter.

We observe that each maximal chain is an induced path. Hence we may define similar properties in the sense of Z.

Definition 5. A vertex v is *doubly Z-terminal* if there is an optimal chronological list such that v is the only vertex in the maximal chain who passes through v. A vertex v is *simply Z-terminal* if there is an optimal chronological list such that v is an endpoint of the maximal chain who passes through v and v is not doubly Z-terminal.

Definition 6. A *reversal* of a zero forcing set F is the set of the last vertices of the maximal chains of a chronological list.

It was shown in [2] that the reversal of a zero forcing set is again a zero forcing set of the same size. In these terminologies, we may rewrite some known results.

Theorem 3 ([8]). *If v is a vertex of G, then we have the following properties.*

- $-1 \leq z_v(G) \leq 1$.
- *The vertex v is doubly Z-terminal if and only if $z_v(G) = 1$.*
- *If v is simply Z-terminal, then $z_v(G) = 0$.*

In order to obtain an exact formula for the zero forcing spread on the vertex-sum, we first need a lemma as follows.

Lemma 4. *If $G = G_1 \oplus_v G_2$, then*

$$Z(G_1) + Z(G_2) - 1 \leq Z(G) \leq Z(G_1) + Z(G_2 - v).$$

Proof. Suppose F is an optimal zero forcing set of G. Denote $F_1 = F \cap V(G_1)$ and $F_2 = F \cap V(G_2)$. In the corresponding chronological list, vertex v is contained in some maximal chain

$$C = v_1 \to v_2 \to \cdots \to v_k.$$

Without loss of generality, we may assume that $v_1 \in G_1$ and $v_i = v$. If C is lying on $V(G_1)$ entirely, we know that F_1 forces $V(G_1)$ and $F_2 \cup \{v\}$ forces $V(G_2)$. Then we have

$$Z(G_1) + Z(G_2) \leq |F_1| + |F_2 + v| \leq Z(G) + 1.$$

Similarly, the inequality holds if C is lying on $V(G_2)$ entirely. Hence we may assume that $C \cap (V(G_i) - v) \neq \varnothing$ for $i = 1, 2$. But in this case we know that F_1 forces $V(G_1)$ and $F_2 \cup \{v\}$ forces $V(G_2)$. The inequality holds again.

On the other hand, suppose F_1 and F_2 are optimal zero forcing sets of G_1 and $G_2 - v$, respectively. Then the second inequality holds by the fact that $F_1 \cup F_2$ is a zero forcing set of G. □

The lemma is useful for deducing the reduction formula for the zero forcing number below.

Theorem 5. *If $G = G_1 \oplus_v G_2$, then the zero spread of G at v is*

$$z_v(G) = \begin{cases} -1 & \text{if } v \text{ is simply Z-terminal of } G_1 \text{ and } G_2; \\ \min\{z_v(G_1), z_v(G_2)\} & \text{otherwise.} \end{cases}$$

Proof. If v is simply Z-terminal in G_1 and G_2, then $z_v(G_1) = z_v(G_2) = 0$ by Theorem 3. Let F_1 and F_2 be optimal zero forcing sets of G_1 and G_2, respectively. We may assume that $v \notin F_1$ and $v \in F_2$ by taking a reversal of F_1 or F_2 if necessary. Thus, $F_1 \cup (F_2 \setminus \{v\})$ forces $V(G)$. Hence,

$$Z(G) \leq Z(G_1) + Z(G_2) - 1.$$

By Lemma 4, $z_v(G) \leq \min\{z_v(G_1), z_v(G_2)\} - 1 = -1$. By Theorem 3, $z_v(G) \geq -1$ and so $z_v(G) = -1$.

Now consider the case when v is not simply Z-terminal in G_1 or not in G_2. Suppose to the contrary that $z_v(G) \neq \min\{z_v(G_1), z_v(G_2)\}$. Since $G - v$ is disconnected with two parts $G_1 - v$ and $G_2 - v$, we have

$$Z(G - v) = Z(G_1 - v) + Z(G_2 - v).$$

By Lemma 4,

$$z_v(G_1) + z_v(G_2) - 1 \leq z_v(G) \leq \min\{z_v(G_1), z_v(G_2)\}.$$

By Theorem 3, the zero spread takes value only in $\{-1, 0, 1\}$. So the only possibility is that $z_v(G) = -1$ and $z_v(G_1) = z_v(G_2) = 0$. This gives that

$$z_v(G_1) + z_v(G_2) = z_v(G) + 1 \text{ or equivalently } Z(G_1) + Z(G_2) = Z(G) + 1.$$

Let F be an optimal zero forcing set of G and

$$C = v_1 \to v_2 \to \cdots \to v_k$$

be the maximal chain containing v. Since $z_v(G) = -1$, the vertex v is neither simply Z-terminal nor doubly Z-terminal of G by Theorem 3. This means that $C \cap (V(G_i) - v) \neq \varnothing$ for $i = 1, 2$. Denote that $F_1 = F \cap V(G_1)$ and $F_2 = F \cap V(G_2)$ and assume that $v_1 \in V(G_1)$. Thus, F_1 forces $V(G_1)$ and $F_2 \cup \{v\}$ forces $V(G_2)$. Then we have the inequality

$$Z(G_1) + Z(G_2) \leq |F_1| + |F_2 + v| = Z(G) + 1,$$

which in fact an equality, and that $F_1, F_2 \cup \{v\}$ are optimal zero forcing sets of G_1 and G_2. This implies that v is simply Z-terminal of both G_1 and G_2, a contradiction. □

Consequently, we have the following results for special cases.

Corollary 6. *If G_1 is a graph with a vertex of degree 1 labeled by v and $G_2 = P_2$ is a path of two vertices with one vertex labeled by v, then*

$$M(G_1 \oplus_v G_2) = M(G_1).$$

The equality also holds when M is replaced by Z or P.

Corollary 7. *If G_1 is a graph with one vertex labeled by v and $G_2 = K_{1,t}$ is a star with $t \geq 2$ and whose center is labeled by v, then*

$$M(G_1 \oplus_v G_2) = M(G_1 - v) + (t - 1).$$

The equality also holds when M is replaced by Z or P.

Corollary 8. *If T is a tree, then*

$$M(T) = P(T) = Z(T).$$

3 Strong PZ condition

In this section, we study the relation between path cover number and zero forcing number. If $P(G) = Z(G)$ for a graph G, we say that G satisfies the *PZ condition*. The following examples show that the induced subgraphs of a graph G (respectively, the vertex-sum of two graphs G_1 and G_2) may not satisfy the PZ condition even if G satisfies (respectively, G_1 and G_2 satisfy) the PZ condition.

Example 9. Let G be the graph in Figure 1. We know that $P(G) = 3 = Z(G)$. But the induced subgraph K_4 does not satisfy the PZ condition, since $P(K_4) = 2$ and $Z(K_4) = 3$.

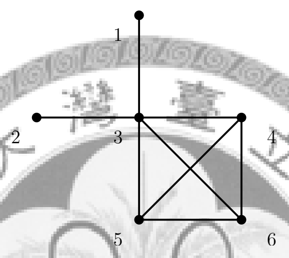

Figure 1: The graph G for Example 9.

Example 10. Let G_1 and G_2 be the graphs in Figure 2 with some vertices labeled v. We have $P(G_1) = Z(G_1) = 1$ and $P(G_2) = Z(G_2) = 3$. But the graph $G = G_1 \oplus_v G_2$ does not satisfy the PZ condition, since $P(G) = 3$ and $Z(G) = 4$.

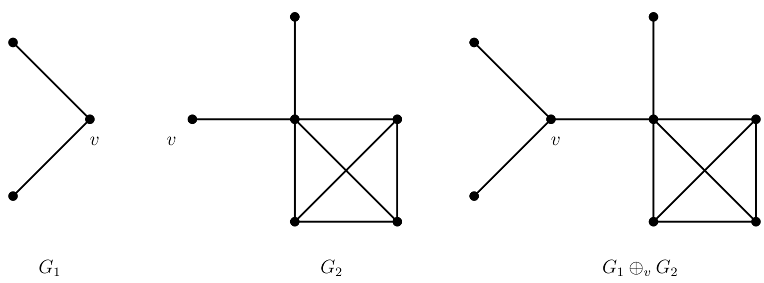

Figure 2: The vertex-sum $G_1 \oplus G_2$ of graphs G_1 and G_2 for Example 10.

This shows that even if the reduction formulae of P and Z are almost the same, they may take different values in vertex-sum operation. So we introduce a stronger condition such that it is hereditary for induced subgraphs and preserved in vertex-sum operation.

Definition 7. For a graph G, if any path cover of G is a set of maximal chains for some zero forcing set, we say that G satisfies the *strong PZ condition.*

It is easy to see that the strong PZ condition implies that $P(G) \geq Z(G)$. And we know that $P(G) \leq Z(G)$ for all graphs G. Hence the strong PZ condition implies the PZ condition. The graph G in Example 9 also provides an example for which the strong PZ condition is really stronger than the PZ condition since the path cover

$$\{\{1\},\{2,3,4\},\{5,6\}\}$$

could not be a set of maximal chains for any zero forcing set.

Also, if H is an induced subgraph of G, each path cover of H is a subset of some path cover of G. If G satisfies the strong PZ condition, then that path cover of G is a set of maximal chains for some zero forcing set of G. Thus, the restriction of this set of maximal chains on $V(H)$ is a set of maximal chains for some zero forcing set of H. Hence, if G satisfies the strong PZ condition, then all its induced subgraphs also satisfy the strong PZ condition.

The following theorem shows that the strong PZ condition holds for graphs composited by smaller graphs satisfying the strong PZ condition.

Theorem 11. *The vertex-sum of two graphs satisfying the strong PZ condition also satisfies the strong PZ condition.*

Proof. Let $G = G_1 \oplus_v G_2$ and Φ be a path cover of G. We have that v is a vertex in some path $P \in \Phi$. Denote

$$P = v_1 - v_2 - \cdots - v_k$$

and $v = v_i$. If all vertices of P falls in $V(G_1)$, define Φ_1 to be the set of those paths in G_1 and Φ_2 to be the set containing those paths in $G_2 - v$ and the one-vertex path v. By the definition of strong PZ condition, Φ_1 and Φ_2 are the sets of maximal chains for some zero forcing set F_1 and F_2 of G_1 and G_2, respectively. So, $\Phi = \Phi_1 \cup (\Phi_2 - v)$ is the set of maximal chains for the zero forcing set $F_1 \cup (F_2 - v)$ of G. Similarly, it works for the case when all vertices of P fall in $V(G_2)$.

Now we may assume that the path P contains vertices in $G_1 - v$ and vertices in $G_2 - v$. Define Φ_1 to be the set of paths entirely lying in G_1 and the path $v_1 - v_2 - \cdots - v_i$, and Φ_2 to be the set of paths entirely lying in G_2 and the path $v_i - v_{i+1} - \cdots - v_k$. Similarly, Φ_1 and Φ_2 are the sets of maximal chains for some zero forcing sets F_1 and F_2 of G_1 and G_2, respectively. Take the reversal if it is necessary. We may assume that $v_1 \in F_1$ and $v_i \in F_2$. Thus, Φ is a set of maximal chains for the zero forcing set $F_1 \cup (F_2 - v)$ of G. This completes the proof of the theorem. □

Since the n-cycle C_n and the 2-path P_2 satisfy the strong PZ condition, we have the following corollary. Recall that a *cactus* is a graph and each of its blocks is a cycle or a K_2.

Corollary 12. *Every cactus G satisfies the strong PZ condition and so $P(G) = Z(G)$.*

4 Graphs with large $Z(G) - M(G)$

Although the path cover number and the zero forcing number is always the same for cactuses. Usually the zero forcing number and the maximum nullity may not be the same for cactuses.

Example 13. The cactus H_5 in Figure 3 is called a 5-*sun*. Notice that $P(G) = Z(G) = 3$ but $M(G) = 2$.

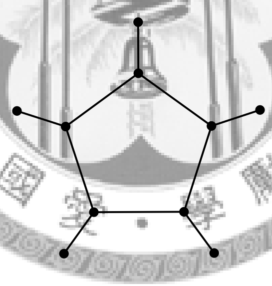

Figure 3: The 5-sun H_5.

In fact, the difference $Z(G) - M(G)$ could be arbitrarily large.

Example 14. The graph in Figure 4 is called a *sequence of* 5-*suns* [5, Fig. 8.]. Denote the sequence of k 5-suns by G_k. It was shown that $P(G_k) - M(G_k) = k$. By Theorem 12, $Z - M$ could be arbitrarily large. More precisely, we may compute those parameters of G_k by using the reduction formulae to get

$$Z(G_k) = P(G_k) = 2k + 1, \ M(G_k) = k + 1.$$

We may modify these graphs to get further results.

Theorem 15. *For any integers p and q with $1 \leq p \leq q \leq 2p - 1$, there is a graph G such that $M(G) = p$ and $Z(G) = q$.*

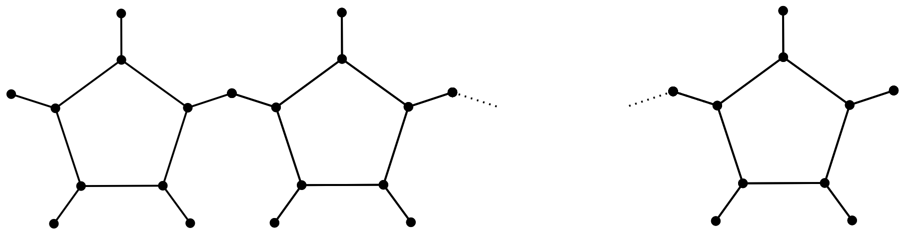

Figure 4: A sequence of k 5-suns G_k.

Proof. If $p = 1$, then $q = 1$ and so we may choose $G = K_1$ as $M(K_1) = Z(K_1) = 1$. We now assume $p \geq 2$. If $q = 2p - 1$, then we choose G to be a sequence of $(p-1)$ 5-sun G_{p-1} as $M(G_{p-1}) = p$ and $Z(G_{p-1}) = 2p - 1 = q$. For the case when $q \leq 2p - 2$, let $k = q - p$ and $t = 2p - q$. Pick a vertex u of degree 1 in G_k. Suppose G'_k is the graph obtained from G_k by adding a leaf v adjacent to u. Also label the center of a star $K_{1,t}$ by v. Since $t \geq 2$,

$$M(G'_k \oplus_v K_{1,t}) = M(G'_k) + (t-1) = M(G_k) + t - 1 = k + t = p$$

$$Z(G'_k \oplus_v K_{1,t}) = Z(G'_k) + (t-1) = Z(G_k) + t - 1 = 2k + t = q,$$

by Lemmas 6 and 7. $\square$

We close this section by the following problem.

Question 16. Does the inequality

$$Z(G) \leq 2M(G) - 1$$

hold for any graph G?

5 Minimum rank of a pattern matrix

We now consider a more general setting for studying the minimum rank of a graph as described below.

Definition 8. The *sign set* S is the set $\{0, *, u\}$. We say a real number r *matches* $0 \in S$ if $r = 0 \in \mathbb{R}$, $* \in S$ if $r \neq 0 \in \mathbb{R}$, and u if r matches $0 \in S$ or $* \in S$. A *pattern matrix* is a matrix whose entries are elements in S.[2] We say that one matrix A over

[2]The "zero-nonzero pattern matrix" defined in Section 9 is a special case of pattern matrices here since the element $u \in S$ is allowed in a pattern matrix. So in this thesis, a pattern matrix, without additional description, means the latter one.

real number is of pattern of a pattern matrix Q, denoted by $A \cong Q$, if they have the same size and each entry in A matches the corresponding entry in Q. The *minimun rank* of a pattern matrix Q is written as $\mathrm{mr}(Q)$ and defined by the smallest rank attained by those matrices A with $A \cong Q$.

We now see an example for the minimum rank of a pattern matrix.

Example 17. Let

$$Q = \begin{pmatrix} * & 0 & 0 \\ u & * & u \end{pmatrix}$$

be a pattern matrix. We observe that the first row and the second row must be linearly independent for any matrix A with $A \cong Q$. So, $\mathrm{mr}(Q) = 2$.

Next, we shall view the rows of a pattern matrix as a "vector" with entries in S. To realize the concept of rank for a pattern matrix, we give the following definitions by simulating the concepts on real vector spaces.

Definition 9. In the sign set $S = \{0, *, u\}$, the addition "+" and the multiplication "×" are defined as follows.

$$+ \colon S \times S \to S$$

+	0	*	u
0	0	*	u
*	*	u	u
u	u	u	u

$$\times \colon \{0, *\} \times S \to S$$

×	0	*	u
0	0	0	0
*	0	*	u

The *sign space of dimension* n is $S^n = \{(x_1, x_2, \ldots, x_n) \colon x_i \in S\}$ with entry-wise addition and scalar multiplication. The elements in S^n are called *sign vectors*.

Notice that the sign space is neither a vector space, a module, nor a matroid. But it still has the distributive law. Since we did not define the multiplication of two scalars, the commutative and the associative laws make no senses. Also, the set has no identity and hence no inverse element for each element.

Definition 10. We say a sign vector $v \sim 0$ if the entries of v contains no $*$. That is, they could only be 0 or u. A finite set of sign vectors $\{v_1, v_2, \dots, v_n\}$ is *linearly independent* (in the sense of sign pattern) if

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n \sim 0,$$

where c_i are scalars in $\{0, *\}$, implies

$$c_1 = c_2 = \cdots = c_n = 0.$$

The *rank* of a set of sign vectors is the maximum cardinality of a independent subset. The *rank* of a pattern matrix is the rank of the set of row vectors of the pattern matrix.

The next lemma shows the relation between the linear independence in the sense of sign pattern and that in the sense of vector space.

Lemma 18. *Suppose $V = \{v_1, v_2, \dots, v_n\}$ is a set of sign vectors, and $W = \{w_1, w_2, \dots, w_n\}$ is a set of sign vectors such that w_i is obtained from v_i by replacing entries u by 0 or $*$. If V is linearly independent, then so is W.*

Suppose $R = \{r_1, r_2, \dots, r_n\}$ is a set of real vectors such that each entry in each vector matches the corresponding entry in elements of W. If W is linearly independent, then R is linearly independent as real vectors.

Proof. To show the first part, it's enough to show that

$$c_1w_1 + c_2w_2 + \cdots + c_nw_n \sim 0$$

implies

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n \sim 0.$$

Since the addition and the scalar multiplication is entry-wise. We only need to prove the case of dimension 1. If $v_t = u$ for some t, by the hypothesis we know that w_t is 0 or $*$. If $c_t = 0$, then $c_tv_t = c_tw_t = 0$ and so there is no difference between $\sum c_iv_i$ and $\sum c_iw_i$. If $c_t = *$, then $\sum c_iv_i = u \sim 0$, since $c_tv_t = u$.

For the second part, similarly it's enough to show that

$$c_1r_1 + c_2r_2 + \cdots + c_nr_n = 0$$

implies

$$c'_1w_1 + c'_2w_2 + \cdots + c'_nw_n \sim 0,$$

where $c'_i = 0 \in S$ if $c_i = 0 \in \mathbb{R}$ and $c'_i = * \in S$ if $c_i \neq 0 \in \mathbb{R}$. Again, we only consider the case of dimension 1. First, if $c_i r_i = 0$ for all i, then $c'_i w_i = 0$ for all i and hence $\sum c'_i w_i = 0 \sim 0$. Second, it's impossible that $\{c_i r_i\}_i$ contains only one nonzero term. So we may assume that $c_s r_s$ and $c_t r_t$ are nonzero for some index s and t. This means that c_s, c_t, r_s and r_t are not zero and hence $c'_s = c'_t = *$ and $w_s = w_t = *$. Hence we have $\sum c'_i w_i = u \sim 0$ since

$$c'_s w_s + c'_t w_t = * + * = u.$$

□

Theorem 19. *If Q is a pattern matrix and U is the set of all pattern matrices obtained from Q by replacing u by 0 or $*$, then*

$$\operatorname{rank}(Q) \leq \min_{Q' \in U}\{\operatorname{rank}(Q')\} \leq \operatorname{mr}(Q).$$

Proof. This is the instant result of Lemma 18 and the definition of rank. □

The integer $\min_{Q' \in U}\{\operatorname{rank}(Q')\}$ is called the *exhaustive rank* of Q, denoted by $\operatorname{erank}(Q)$.

6 Rank of pattern matrix vs zero forcing number

The concept rank of a pattern matrix maybe is a little bit abstract. However, by defining a general zero forcing process, we can get an interpretation of the rank of a pattern matrix.

Definition 11. Suppose $G = (V, E)$ is a graph and B is a subset of E called the set of *banned edges* or the *banned set* for short. The *zero forcing process on a graph G banned by B* is the coloring process by following rules.

- Each vertex of G is either black or white initially.
- If x is a black vertex and y is the only white neighbor of x and $xy \notin B$, then change the color of y to black.

If F is a subset of V and by using F as the initial set of black vertices we can change all vertices of G to black by the zero forcing process banned by B, then F is called a *zero forcing set of G banned by B*. The *zero forcing number of G banned by B* is denoted by $Z(G, B)$ and defined by the minimum cardinality of a zero forcing set F banned by B.

Let W be a subset of V. A subset $F \subseteq V$ is a *zero forcing set banned by B with support W* means that F is a zero forcing set banned by B and $W \subseteq F$. The *zero forcing number of G banned by B with support W* is the minimum cardinality of F such that F is a zero forcing set banned by B with support W. This number is denoted by $Z_W(G, B)$.

It's clear that if B and W are the empty set, then the process has no difference with the original zero forcing process.

Now with this definition, for any $m \times n$ pattern matrix Q we can construct a graph and a set of banned edges such that the zero forcing number of the graph is the value $m + n - \text{rank}(Q)$. For Q, consider the bipartite $G = (X \cup Y, E)$ with

$$X = \{a_1, a_2, \ldots, a_m\},\ Y = \{b_1, b_2, \ldots, b_n\},\ E = \{a_i b_j : Q_{ij} \neq 0\}.$$

Also let the set of banned edges to be

$$B = \{a_i b_j : Q_{ij} = u\}.$$

We then have the following relation between the rank and the zero forcing number.

Theorem 20. *For a given $m \times n$ pattern matrix Q, If $G = (X \cup Y, E)$ and B are the graph and the set of banned edges defined above, then*

$$\text{rank}(Q) + Z_Y(G, B) = m + n.$$

Proof. Denote $[m]$ for the set $\{1, 2, \ldots, m\}$. Let $\{v_i\}_{i \in [m]}$ be the rows of Q. For any $I \subseteq [m]$, we want to show that $\{v_i\}_{i \in I}$ is independent if and only if $Y \cup \{a_i\}_{i \in [m] \setminus I}$ is a zero forcing set banned by B with support Y.

For the sufficiency, suppose $Y \cup \{a_i\}_{i \in [m] \setminus I}$ is a zero forcing set banned by B using $\{a_i\}_{i \in I}$ as the initial set of white vertices. Suppose $a_s \in X$ is the first white vertex who was changed color into black by some vertex $b_t \in Y$. Then a_s is the only white neighbor of b_t and $a_s b_t \notin B$. This means that among all the vectors in $\{v_i\}_{i \in I}$, the sign vector v_s is the only one whose t-th entry is $*$ and all other vectors have their t-th entries zero. So, if

$$\sum_{i \in I} c_i v_i \sim 0,$$

then c_s must be zero otherwise the t-th entry of that sum would be $*$. Since every white vertex will be changed into black, every c_i is forced to be zero.

For the necessity, suppose $\{v_i\}_{i\in I}$ is independent. Initially, set all vertices in $\{a_i\}_{i\in I}$ to be white and others to be black. Write these vectors and the hypothesis of independence as

$$v_i = (x_{i1}, x_{i2}, \dots, x_{in})$$

and

$$\sum_{i\in I} c_i x_{ij} \sim 0,$$

for all $j = 0, 1, \dots, n$. Now, if for every j the set $\{x_{ij}\}_{i\in I}$ contains u or two more $*$ or all the elements are zero, then $c_i = *$ for all $i \in I$ is a nontrivial solution and thus the set is not independent. So there must exist an integer t such that $\{x_{it}\}_{i\in I}$ contains exactly one $*$, say $x_{st} = *$, and all other elements are zero. This means b_t has only one white neighbor a_s and $a_s b_t \notin B$. So we can force a_s to be black. Doing this inductively, every white vertex will be forced into black. $\square$

Example 21. We already know that the rank of

$$Q = \begin{pmatrix} * & 0 & 0 \\ u & * & u \end{pmatrix}$$

is 2. The graph described by Q is given in Figure 5 with $Z_Y(G,B) = 3$ and

$$\operatorname{rank}(Q) + Z_Y(G,B) = 5 = m + n.$$

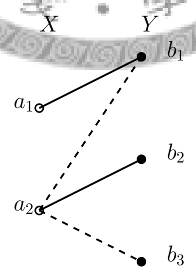

Figure 5: The graph G described by a pattern Q.

The proof indicates some reasons for the name "zero forcing" process since the process forces the coefficients to be zero one by one.

On the other hand, if we call the rank we defined above the "row rank" and the number of maximum independent column sign vectors the "column rank". Are the two values the same just as that in general vector spaces? The answer is yes. And we can prove it in language of graph theory.

Theorem 22. *With the same condition in Theorem 20, we have*

$$Z_Y(G,B) = Z_X(G,B).$$

Proof. Since the reversal[3] of a zero forcing set is also a zero forcing set, we know the two values are the same. □

Combining Theorems 20 and 22, we know that the "row rank" will always equal to the "column rank" in any pattern matrix.

7 Exhaustive zero forcing number of graphs

By Theorem 19, there is some parameter between the rank and the minimum rank of a pattern matrix. We can use this tool to give a better bound for the maximum nullity of a graph. Here is a simple example.

Example 23. The corresponding pattern matrix of the 3-path P_3 is

$$Q = \begin{pmatrix} u & * & 0 \\ * & u & * \\ 0 & * & u \end{pmatrix}.$$

Since there is one more condition, symmetry, in the minimum rank problem,

$$\mathrm{mr}(P_3) \geq \mathrm{mr}(Q) \geq \mathrm{erank}(Q) \geq \mathrm{rank}(Q).$$

And all the values of those parameter are 2 here.

Just as in the example, for a graph $G = (V,E)$ we have a corresponding pattern matrix Q with

$$Q_{ij} = Q_{ji} = \begin{cases} u, & \text{if } i = j; \\ *, & \text{if } i \neq j, ij \in E; \\ 0, & \text{if } i \neq j, ij \notin E. \end{cases}$$

This gives an instant proposition.

Proposition 24. *If G is a graph and Q is the corresponding pattern matrix, then*

$$\mathrm{mr}(G) \geq \mathrm{mr}(Q) \geq \mathrm{erank}(Q) \geq \mathrm{rank}(Q).$$

[3]Although our zero forcing process is different from the general process, those banned edges will not disturb the proof in paper [2, Theorem 2.6].

Furthermore, let U be the set of pattern matrices obtained from Q by replacing the entries equal to u by 0 or $*$. So, the set U contains 2^n pattern matrices, where n is the number of vertices of G. Let $[n] = \{1,2,\dots n\}$ and I be a subset of $[n]$. Construct a family of bipartite graphs $\widetilde{G}_I$ corresponding to elements in U with

$$\text{vertex set } V(\widetilde{G}_I) = X \cup Y, \text{ where } X = \{a_1, a_2, \dots, a_n\} \text{ and } Y = \{b_1, b_2, \dots, b_n\},$$

$$\text{and edge set } E(\widetilde{G}_I) = \{a_i b_j : i \neq j, ij \in E(G)\} \cup \{a_i b_i : i \in I\}.$$

By the arguments of the last two sections,

$$\operatorname{erank}(Q) + \max_{I \subseteq [n]} Z_Y(\widetilde{G}_I) = 2n$$

and

$$\operatorname{rank}(Q) + Z_Y(\widetilde{G}_{[n]}, B) = 2n,$$

where the set of banned edges B is $\{a_i b_i : i \in [n]\}$. Hence we can change the minimum rank problem into the maximum nullity problem and get

$$M(G) \leq \max_{I \subseteq [n]} Z_Y(\widetilde{G}_I) - n \leq Z_Y(\widetilde{G}_{[n]}, B) - n.$$

We call the second term in the inequality above the *exhaustive zero forcing number* and denote it by $\widetilde{Z}(G)$.

Example 25. Let Q be the pattern matrix given by graph P_3. To find the exhaustive zero forcing number of Q, we need compute the zero forcing number of the 8 graphs in Figure 6. The number written below a graph is the number $Z_Y(\widetilde{G}_I)$.

Now we have three upper bounds $Z(G)$, $\widetilde{Z}(G)$ and $Z_Y(\widetilde{G}_{[n]}, B) - n$ for maximum nullity of a graph. To compare these bounds, we give the following theorem.

Theorem 26. *For a graph G, let $\widetilde{G}_{[n]}$ and B be the graph and the set of banned edges defined above. Then, $J \subseteq [n] = V(G)$ is a zero forcing set of G if and only if $Y \cup \{a_i\}_{i \in J}$ is a zero forcing set of $\widetilde{G}_{[n]}$ banned by B. Consequently,*

$$M(G) \leq \widetilde{Z}(G) \leq Z(G).$$

Proof. Observe that v_i forces v_j in the chronological list on G if and only if b_i forces a_j in the chronological list on $G_{[n]}$. So the theorem comes from the fact that they follow the corresponding chronological list. □

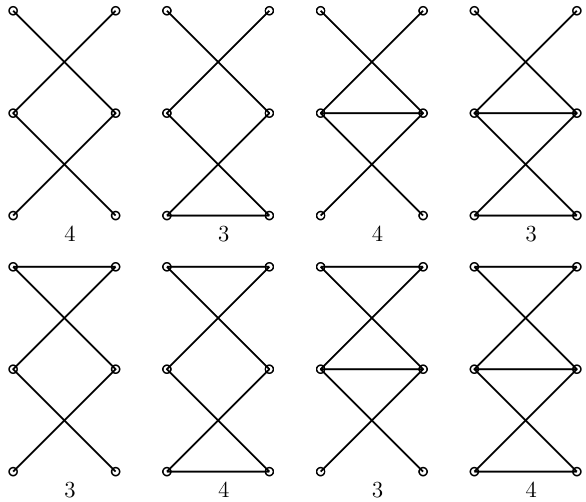

Figure 6: The 8 graphs used in computing exhaustive zero forcing number.

Thus we know that $\widetilde{Z}(G)$ is a better upper bound than $Z(G)$ for $M(G)$. Furthermore, by Theorem 26, if $M(G) = Z(G)$ then $\widetilde{Z}(G)$ also takes that same value.

Finally, we count the exhaustive zero forcing number of a n-sun H_n with n odd to show that $\widetilde{Z}(G)$ and $Z(G)$ may be surely different sometimes. The value $Z(H_n)$ and $M(H_n)$ used in the theorem comes from paper [4] and the fact $P(G) \leq Z(G)$.

Theorem 27. *If H_n is the graph obtained from the n-cycle C_n by adding a leaf to each vertex on C_n, then*

$$\widetilde{Z}(H_n) = \begin{cases} 2, & \text{if } n = 3; \\ \lfloor \frac{n}{2} \rfloor & \text{if } n > 3. \end{cases}$$

Proof. We already have that

$$M(H_3) = Z(H_3) = 2$$

and

$$M(H_n) = Z(H_n) = \frac{n}{2} = \lfloor \frac{n}{2} \rfloor$$

when n is even. So, by the inequality in Theorem 26, the statement is true for these two cases. We then only need to show that

$$\widetilde{Z}(H_n) = \lfloor \frac{n}{2} \rfloor$$

when n is odd and $n \geq 5$. However, since in this case we already have

$$\lfloor \frac{n}{2} \rfloor = M(H_n) \leq \widetilde{Z}(H_n) \leq Z(H_n) = \lceil \frac{n}{2} \rceil,$$

it's enough to show that $\widetilde{Z}(H_n) \neq \lceil \frac{n}{2} \rceil$.

For brevity, we write $G = H_n$. We label the vertices on the subgraph C_n $h_1, h_2, \ldots h_n$ counterclockwise, and the leaf adjacent to h_i by h_i'. Set I be a subset of

$$\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$$

and

$$I' = I \cap \{1', 2', \ldots, n'\}.$$

Now for any fixed I, we want to show that $Z_Y(\widetilde{G}_I) - n < \lceil \frac{n}{2} \rceil$ by constructing a zero forcing set with cardinality less than that number. Similarly we write the corresponding vertices in X as a_i and a_i', while the corresponding vertices in Y as b_i and b_i'. And we have the following coloring rules.

1. b_i' forces a_i if $i' \notin I'$ or a_i' is black.
2. b_i forces a_{i+1} if a_{i-1}, a_i, and a_i' are black.[4]
3. b_i' forces a_i' if a_i is black and $i' \in I'$.
4. Each a_i' will be forced if each a_i is black.

Now we consider two cases.

Case 1. If $I' = \{1', 2', \ldots, n'\}$, then we take $Y \cup \{a_1', a_2'\}$ as a zero forcing set. First a_1 and a_2 are forced by rule 1. Second a_3 is forced by rule 2. Finally a_3' is forced by rule 3 and then a_4 is forced by rule 2 again. Continuing the final step and each a_i will be forced and then we can use rule 4.

Case 2. If $I' \neq \{1', 2', \ldots, n'\}$, then I' can be partitioned into several intervals in view of circle, say T_k. Let F be the set consisting of each a_i' such that i' is the first element in each T_k in the order of $1', 2', \ldots, n', 1'$. We say $Y \cup F$ is a zero forcing set. Without loss of generality we may assume T_1 begin by $1'$. First a_1 and each vertex a_i will be forced if $i' \notin I$ by rule 1. Second a_2 will be forced by rule 2. Finally a_2' is forced by rule 3 and then a_3 is forced by rule 2 again. Continuing the final step and each a_i with $i' \in T_1$ will be forced. Also doing this process to each T_k, then each vertex a_i will be forced and then we can use rule 4.

In the Case 2 we know that k has value at most $\lfloor \frac{n}{2} \rfloor$. So,

$$\widetilde{Z}(H_n) \leq \max\{2, \lfloor \frac{n}{2} \rfloor\} = \lfloor \frac{n}{2} \rfloor,$$

since $n \geq 5$. This completes the proof of the theorem. □

[4] Our addition takes modulus of n.

The fact that the exhaustive zero forcing number is better than the zero forcing number is a little bit surprising. It is natural that an unknown element u could be zero or nonzero. But it is surprising that when we consider all cases that u might be, the bound is strictly sharper. The reason is that we give too much possibility when we use the element u. We might see the detail in graphs. Suppose i and j are two vertices in a graph G. When i and j are adjacent, we might force j by i but this may simultaneously increase the number of the white neighbors of i. When i and j are not adjacent, although we cannot force j by i, the number of white neighbors of i is relatively small. So in general the two value $Z(G)$ and $Z(G-e)$ are not comparable. However, a banned edge take both the disadvantages of an edge and a non-edge. So no matter i and j are adjacent or not, we must have

$$Z(G,B) \le Z(G+e,B+e),$$

where $e = ij$.

Now with this new parameter, the question is, could the equality $\widetilde{Z}(G) = M(G)$ holds for all cactus graph G? The following example gives a negative answer.

Example 28. Let G be the graph in Figure 7. We have $M(G) = 2$. But by some computation and by setting $I = \{1, 2, \ldots, 15\}$, we have $\widetilde{Z}(G) = 3$.

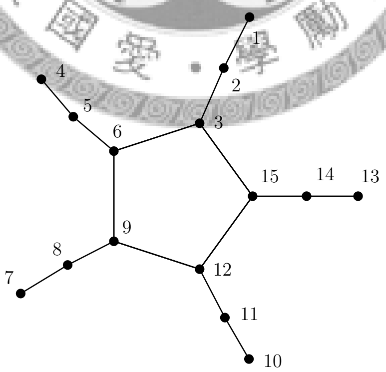

Figure 7: A graph G for Example 28.

8 Sieving process

In most of time, the computation of the exhaustive zero forcing number is tedious. For example, to determine the value $\widetilde{Z}(G)$ for some graph G with n vertices, the maximum in the definition should run over 2^n cases. It increases rapidly. However, we may rewrite the definition as

$$\widetilde{Z}(G) = \max_{I\subseteq[n]} Z_Y(\widetilde{G}_I) - n = \max\{k: k = Z_Y(\widetilde{G}_I) - n \text{ for some } I\}.$$

So, for each integer k, we may define $\mathcal{I}_k$, or $\mathcal{I}_k(G)$ if it is necessary to mention the graph, to be the set of possible indices such that

$$Z_Y(\widetilde{G}_I) - n \geq k.$$

Again we may rewrite the value as

$$\widetilde{Z}(G) = \max\{k: \mathcal{I}_k \neq \varnothing\}.$$

On the other hand, to determine whether I is in $\mathcal{I}_k$ or not, we should check that each subset $F \supseteq Y$ with size $n + k - 1$ cannot force $V(\widetilde{G}_I)$. Suppose now $F \supseteq Y$ is a subset with size $n + k - 1$. Thus we get a set of candidates for $\mathcal{I}_k$. That is, the set F cannot force $V(\widetilde{G}_I)$ for those candidates I. So each subset $F \supseteq Y$ with size $n + k - 1$ is like a sieve who only allows those candidates to pass. And the set $\mathcal{I}_k$ is the remaining set after all indices was sieved by each $F \supseteq Y$ with size $n + k - 1$. Hence we call this process the *sieving process*.

Example 29. Label the vertices of a 5-sun H_5 in Figure 8. We already know that $M(H_5) = 2$ but $Z(H_5) = 3$. So the exhaustive zero forcing number could only be 2 or 3. So we want to say $\widetilde{Z}(H_5) = 2$ directly by showing that $\mathcal{I}_3$ is empty. We pick

$$F = \{a_5, a_7\} \cup Y$$

as the first sieve. We know that, no matter what the index I was chosen, b_5, b_7 can force a_6, a_8. Then b_6, b_8 can force a_4, a_{10}. If now $9 \in I$, then b_9 can force a_9. Thus all the vertices would be forced into black. Hence we know that $9 \notin I$. Furthermore, by automorphisms of H_5, we know that $1, 3, 5, 7, 9$ cannot be elements in I if $I \in \mathcal{I}_3$. This consequence already leaves 2^5 possible indices. On the other hand, observe that if $1 \notin I$, then b_1 can force a_2. This will also make all vertices black. So we must have $1 \in I$. Combining $1 \in I$ and $1 \notin I$, we know that $\mathcal{I}_3$ is empty just by only one sieve.

Example 30. Let G_k be the sequence of k 5-suns in Example 14. Label the vertices of G_k as that in Figure 9. We would like to show that $\widetilde{Z}(G_k)$ is no greater than

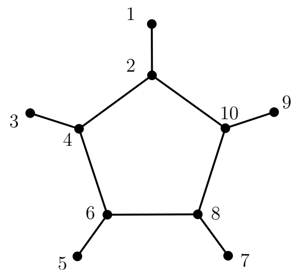

Figure 8: A 5-sun H_5 with labeled vertices.

$k+1$ by the sieving process. If I is an index set, we focus on whether $i(5)$ is an element in I or not and sieve every possibilities to show that $\mathcal{I}_{k+2}(G_k)$ is empty. Let $F = Y$ initially. If $1(5) \in I$, add $a_{1(1)}$ and $a_{1(2)}$ into F. If $1(5) \notin I$, add $a_{1(2)}$ and $a_{1(8)}$ into F. For other $i = 2, 3, \ldots, k$, add $a_{i(2)}$ into F if $i(5) \in I$ while add $a_{i(8)}$ into F if $i(5) \notin I$. Thus the cardinality of $F \backslash Y$ is $k+1$. And for a given index I, it will be sieved by the corresponding F. So the set $\mathcal{I}_{k+2}(G_k)$ is empty. Hence $\widetilde{Z}(G_k) \leq k+1$. With the addition that $M(G_k) = k+1$, the bound $\widetilde{Z}(G_k)$ is sharp.

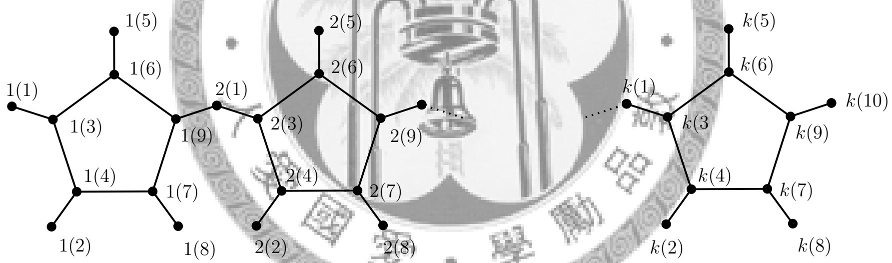

Figure 9: A sequence of k 5-sun G_k with labeled vertices.

The sieving process not only gives a better way to determine the value of the exhaustive zero forcing number, but also gives some further information. Now denote Q to be the pattern matrix given by the graph G while Q_I to be the pattern matrix given by the bipartite graph $\widetilde{G}_I$. Clearly, Q_I is a pattern matrix obtained from Q by replacing some u's with some $*$'s and 0's. Suppose now k is a given integer and A is a given real matrix such that A is of pattern of Q. By Theorem 20, the nullity of A is less than k if A is of pattern of Q_I for $I \notin \mathcal{I}_k$. Or equivalently, if a matrix A of pattern of Q has its nullity greater than or equal to k, then A must be of pattern Q_I for some $I \in \mathcal{I}_k$. Especially, if now $k = M(G)$, the matrix attaining this maximum value k must has the pattern of Q_I for some $I \in \mathcal{I}_k$. This can give some clues on how to find this matrix A.

Example 31. Label the vertices of the bipartite graph $G = K_{1,3}$ in Figure 10. We know that $M(G) = Z(G) = 2$. To find the set I_2, we may pick

$$F = \{a_1\} \cup Y$$

as the first sieve. Similarly, no matter what the index I was chosen, b_1 can force a_2. Now if $3 \in I$, then b_3 can force a_3. Thus all vertices will become black. Hence we have $3 \notin I$ for all $I \in \mathcal{I}_2$. Also, by automorphisms of $K_{1,3}$, we know that $1, 3, 4$ cannot be elements in I. Thus,

$$\mathcal{I}_2 = \{\{2\}, \varnothing\}.$$

This means that if A is a matrix with nullity 2 such that $\mathcal{G}(A) = K_{1,3}$, the ii-entry, $i = 1, 3, 4$, must be zero.

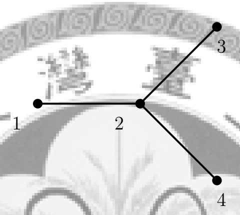

Figure 10: A bipartite graph $K_{1,3}$ with labeled vertices.

Example 32. Let $G = K_{3,3,3}$ be the complete 3-partite $K_{3,3,3}$ with labeled vertices in Figure 11. It is easy to compute $Z(G) = 7$. But amazingly, the only index set in $\mathcal{I}_7(G)$ is the empty set. Let

$$F = \{a_3, a_4, a_5, a_6, a_7, a_8\} \cup Y$$

be the first sieve and I is an index set in $\mathcal{I}_7(G)$. Independent of I, b_3 can force a_9. If 1 is an element in I, then b_1 can force a_1. Thus b_4 can force a_2 and all vertices are forced. Hence 1 cannot be an element in I. By automorphisms of $K_{3,3,3}$, we get $\mathcal{I}_7(G) = \{\varnothing\}$.

However, although $\widetilde{Z}(G) = 7$, it is known that $M(K_{3,3,3}) = 6$ in [6]. Here we give another view of this fact. If a matrix A in $\mathcal{S}(G)$ attains the nullity 7, then the diagonal entries of A must be zero since $\mathcal{I}_7(G) = \{\varnothing\}$. Thus by multiplying some scalar to corresponding columns and rows simultaneously we may write A as the form

$$A = \begin{pmatrix} O & J & J \\ J & O & B^\top \\ J & B & O \end{pmatrix},$$

where O is the zero matrix and J is the matrix whose entries are all 1 with appropriate size. Do the column and row operations simultaneously, the matrix

$$\begin{pmatrix} O & J & O \\ J & O & B^\top \\ O & B & -B-B^\top \end{pmatrix}$$

also has the nullity 7. To attain the nullity 7, the block $-B-B^\top$ must be a zero matrix. This means B is symmetric. Since B is a matrix without zero entries and the characteristic of the field of real numbers is not 2, this kind of B does not exist. Hence $M(G) \leq 6$. And the matrix

$$\begin{pmatrix} O & J & J \\ J & O & J \\ J & J & O \end{pmatrix}$$

attains this nullity.

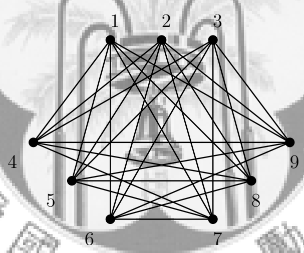

Figure 11: The complete 3-partite $K_{3,3,3}$ with labeled vertices.

As what we did in Example 31, for some fixed integer k, we may find some vertices who cannot be elements in I for all I in $\mathcal{I}_k$. These vertices give those matrices who attained the nullity k a special pattern. It would be a good tool for us to get a better upper bound for the maximum nullity.

Definition 12. Let i be a vertex of a graph G. The vertex i is a *zero-vertex* in $\mathcal{I}_k(G)$ if i is not an element in I for all $I \in \mathcal{I}_k(G)$. And it is a *nonzero-vertex* in $\mathcal{I}_k(G)$ if i is always an element in I for all $I \in \mathcal{I}_k(G)$.

By doing some sieving processes, zero-vertices or nonzero-vertices appear in many graphs. For the complete graph K_n, every vertex is a nonzero-vertex in $\mathcal{I}_{n-1}(K_n)$ if $n \geq 2$ and the vertex in K_1 is a zero-vertex in $\mathcal{I}_1(K_1)$. For the complete bipartite $K_{1,t}$ with $t \geq 2$, each leaf is a zero-vertex in $\mathcal{I}_{t-1}(K_{1,t})$. And for those complete multi-partite graphs G with two more parts and three more vertices in each part, every vertex is a zero-vertex in $\mathcal{I}_{n-2}(G)$, where n is the number of vertices of G. Finally, for the 5-sun H_5 in Figure 3, each leaf is a zero-vertex and a nonzero-vertex in $\mathcal{I}_3(H_5)$ simultaneously. This means that $\mathcal{I}_3(H_5)$ is empty and the exhaustive zero forcing number is no greater than 2.

The advantage of a nonzero-vertex is that we may do the Gaussian elimination on the matrix who attains the considered nullity.

Theorem 33 (Nonzero Elimination Lemma). *For a graph G, suppose i is a nonzero-vertex in $\mathcal{I}_k(G)$ and $\eta_i(G)$ denote the set of those graphs obtained from G by the following rules:*

- *the vertex i should be deleted;*
- *for any neighbors x and y of i, the pair xy should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$.*

If the nullity k is achievable by some matrix in $\mathcal{S}(G)$, then

$$k \leq \max\{M(H) : H \in \eta_i(G)\}.$$

Proof. Let A be a matrix in $\mathcal{S}(G)$ attaining the nullity k. We may assume that i is the first vertex 1. Since 1 is a nonzero-vertex in $\mathcal{I}_k(G)$, the 11-entry of A must be nonzero. Assume the 11-entry is 1 by multiplying some scale to A without change the pattern and the nullity of A. Also, by some permutation we may assume the matrix A is of the form

$$\begin{pmatrix} 1 & a^\top & 0 \\ a & \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix},$$

where a is a vector whose coordinates are all nonzero. By doing row operations and column operations, we may get the new matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{B}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}$$

with the same nullity k, where

$$\widehat{B}_{11} = \widehat{A} - aa^\top.$$

Since $aa^\intercal$ is a square matrix with all entries nonzero, the entries of $\widehat{B}_{11}$ is nonzero if they are zero in $\widehat{A}_{11}$ and unknown otherwise. Let B be the matrix

$$\begin{pmatrix} \widehat{B}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}.$$

Then we have the inequality

$$k = \text{null}(A) = \text{null}(B) \le M(H),$$

for some H in $\eta_i(H)$. Hence

$$k \le \max\{M(H) : H \in \eta_i(G)\}.$$

□

Example 34. Let G be the graph in Figure 7. In Example 28 we know that $M(G) = 2$ but $Z(G) = \widetilde{Z}(G) = 3$. Here give some reasons to it. First we observe that the vertex 1 is a nonzero-vertex in $\mathcal{I}_3(G)$ by following arguments. We may pick the set

$$F = \{a_4, a_7\} \cup Y$$

as the first sieve. Then b_4 and b_7 force a_5 and a_8 in each I. And consecutively, a_6, a_9, a_3, and a_{12} are forced into black. In this situation, if 1 is not an element in I, then b_1 can force a_2, then b_2 forces a_1. Thus all vertices would be forced into black. This means 1 is a nonzero-vertex in $\mathcal{I}_3(G)$. In this case, the graph $G - 1$ is the only graph in the set $\eta_1(G)$. By applying Theorem 33, if the nullity 3 is achievable, then we get

$$3 \le M(G-1) \le \widetilde{Z}(G-1) = 2.$$

This is a contradiction. So no matrix in $\mathcal{S}(G)$ has nullity 3 and thus $M(G) \le 2$. The bound is sharp now.

Example 35. Let G be the graph in Figure 12. In Example 2.11 of paper [2], it was shown that $M(G) = P(G) = 3$ and $Z(G) = 4$. The upper bound 3 could also be given by Theorem 33. The vertex 1 is a nonzero-vertex in $\mathcal{I}_4(G)$ by the sieve

$$F = \{a_{11}, a_{12}, a_3\} \cup Y.$$

Let e be the edge adjoining vertex 2 and vertex 3 in G. Then the set $\eta_1(G)$ contains only two graphs $G - 1$ and $G - 1 - e$. If the nullity 4 is achievable, then by Theorem 33

$$4 \le \max\{M(G-1), M(G-1-e)\} \le \max\{Z(G-1), Z(G-1-e)\} = 3.$$

This is a contradiction. Hence no matrix in $\mathcal{S}(G)$ achieves the nullity 4. This means $M(G) \le 3$.

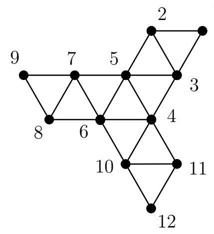

Figure 12: The pinwheel on 12 vertices.

On the other hand, there is still some work we can do on those zero-vertices. If a matrix is of the pattern

$$\begin{pmatrix} 0 & * \\ * & u \end{pmatrix},$$

then the matrix must be invertible and the inverse must be of the pattern

$$\begin{pmatrix} u & * \\ * & 0 \end{pmatrix}.$$

Thus the similar work could be done on a zero-vertex.

Theorem 36 (Zero Elimination Lemma). *For a graph G, suppose i is a zero-vertex in $\mathcal{I}_k(G)$ and j is a neighbor of i. Let*

$$N_1 = \{v \ne j : iv \in E(G)\},\ N_2 = \{v \ne i : jv \in E(G), iv \notin E(G)\}.$$

And $\eta_{i\to j}(G)$ denote the set of those graphs obtained from G by the following rules:

- *the vertex i and j should be deleted;*
- *for $x \in N_1$ and $y \in N_2$, the pair xy should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$;*
- *for x and y in N_1, the pair xy could be an edge or a non-edge.*

If the nullity k is achievable by some matrix in $\mathcal{S}(G)$, then

$$k \le \max\{M(H) : H \in \eta_{i\to j}(G)\}.$$

Proof. Let A be a matrix in $\mathcal{S}(G)$ attaining the nullity k. We may assume that i is the first vertex 1. Since 1 is a zero-vertex in $\mathcal{I}_k(G)$, the 11-entry of A must be zero. By some permutation we may assume the matrix A is of the form

$$\begin{pmatrix} \alpha & a^\top & O \\ a & \widehat{A}_{11} & \widehat{A}_{12} \\ O & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix},$$

where α is a 2×2 matrix of the pattern

$$\begin{pmatrix} 0 & * \\ * & u \end{pmatrix}$$

and a is a two-column matrix with none of its row vector to be a zero vector. Then the matrix

$$\begin{pmatrix} \alpha & O & O \\ O & \widehat{B}_{11} & \widehat{A}_{12} \\ O & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}$$

has the same nullity k, where

$$\widehat{B}_{11} = \widehat{A} - a\alpha^{-1}a^\top.$$

The fact that the 11-entry of A is zero make sure that the matrix α^{-1} exist and is of the pattern

$$\begin{pmatrix} u & * \\ * & 0 \end{pmatrix}.$$

Let B be the matrix

$$\begin{pmatrix} \widehat{B}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}.$$

Observe the xy-entry of the matrix $a\alpha^{-1}a^\top$ is zero if x and y are in N_2, is nonzero if x is in N_1 and y is in N_2, and is unknown if x and y are all in N_1. Thus the graph $\mathcal{G}(B)$ must be a graph in $\eta_{i\to j}(G)$.

Then we have the inequality

$$k = \text{null}(A) = \text{null}(B) \leq M(H),$$

for some H in $\eta_{i\to j}(H)$. Hence

$$k \leq \max\{M(H) : H \in \eta_{i\to j}(G)\}.$$

□

Example 37. Let G be the left graph in Figure 13 called *semimoth*. The graph G is outerplanar. By some discussions, we have $P(G) = Z(G) = 5$. But we would like to show that $M(G) \le 4$ in three ways.

First observe that the vertex 1 is a zero-vertex in $\mathcal{I}_5(G)$ by the sieve

$$F = \{a_8, a_9, a_{13}, a_{14}\} \cup Y.$$

To apply Theorem 36 on vertices $1 \to 2$, we have

$$N_1 = \{3\},\ N_2 = \{7, 13, 15\}.$$

Hence the right graph H in Figure 13 is the only graph in $\eta_{1\to 2}(G)$. The zero forcing number of the graph H is less than or equal to 4 since $\{9, 10, 12, 15\}$ is a zero forcing set. If the nullity 5 is achievable, by Theorem 36, we get the contradiction

$$5 \le Z(H) \le 4.$$

Hence $M(G) \le 4$.

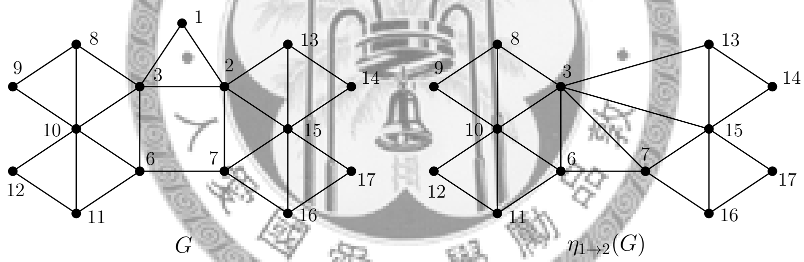

Figure 13: The semimoth G and the set $\eta_{1\to 2}(G)$ for Example 37.

Second observe that the vertex 1 is also a nonzero-vertex in $\mathcal{I}_5(G)$ by the sieve

$$F = \{a_9, a_{10}, a_{12}, a_{15}\}.$$

And the set $\eta_1(G)$ is shown in Figure 14. The left one has zero forcing number less than 4 since $\{8, 9, 13, 14\}$ is a zero forcing set; while the right one has maximum nullity 4 by the reduction formula in Theorem 1. If the nullity 5 is achievable, by Theorem 33, we get the contradiction $5 \le 4$. Hence $M(G) \le 4$.

Finally, since the vertex 1 is a zero-vertex and a nonzero-vertex in $\mathcal{I}_5(G)$ simultaneously. We know that $\widetilde{Z}(G) \le 4$. Therefore we get $M(G) \le 4$ again.

Example 38. Let G be the graph in Figure 15. By some intricate discussions, we have $P(G) = 5$ and hence $Z(G) = 5$. Also, we have $\widetilde{Z}(G) = 5$ since the maximum is

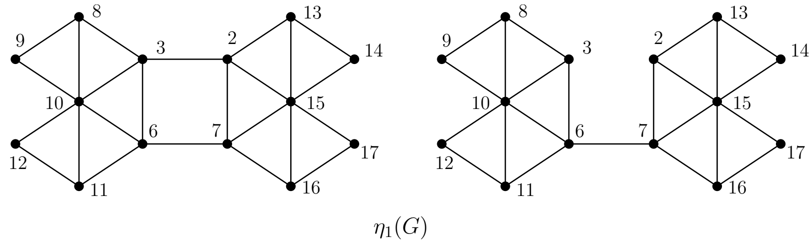

Figure 14: The set $\eta_1(G)$ for Example 37.

attained by the index set $I = V(G)$. However, we would like to illustrate the fact $M(G) \leq 4$.

Obseve that the vertex 1 is a nonzero-vertex in $\mathcal{I}_5(G)$ by the sieve

$$F = \{a_8, a_9, a_{13}, a_{14}\} \cup Y.$$

Furthermore, the semimoth mentioned in Example 37 is the only graph in $\eta_1(G)$. Since we already know the maximum nullity of semimoth is less than or equal to 4, we get the contradiction $5 \leq 4$ by Theorem 33 if the nullity 5 is achievable. Thus we know $M(G) \leq 4$.

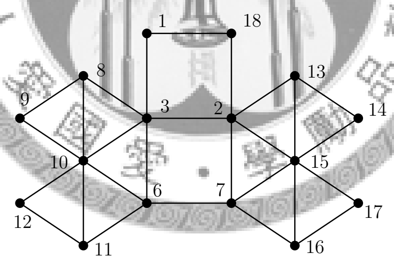

Figure 15: The graph G for Example 38.

These elucidate that we may use Theorems 33 and 36 to get some contradictions. And sometimes we may use them not exactly once. In conclusion, the sieving process help us find the pattern of those matrices with large nullity. And by doing Gaussian elimination, we may get a better upper bound sometimes.

We end this section by the following corollaries.

Corollary 39 (Simple Elimination Lemma)**.** *If i is a vertex of a graph G and j is a neighbor of i, then*

$$M(G) \leq \max\{M(H) \colon H \in \eta_i(G) \cup \eta_{i \to j}(G)\}.$$

Proof. Since the ii-entry of a matrix in $\mathcal{S}(G)$ attained the nullity $M(G)$ could only be zero or nonzero, the inequality holds by Theorems 33 and 36. $\square$

Corollary 40 (Double Zero Elimination Lemma). *For a graph G, suppose i and j are zero-vertices in $\mathcal{I}_k(G)$ and j is a neighbor of i. Let*

$$N_1 = \{v \neq j : iv \in E(G), jv \notin E(G)\},\ \ N_2 = \{v \neq i : jv \in E(G), iv \notin E(G)\}$$

$$\text{and } N_3 = \{v : iv \in E(G), jv \in E(G)\}.$$

Also, let N_0 be the subset of N_3 containing zero-vertices in $\mathcal{I}_k(G)$ and $\eta_{i-j}(G)$ denote the set of those graphs obtained from G by the following rules:

- *the vertex i and j should be deleted;*
- *for x and y in $N_1 \cup N_2 \cup N_3$ such that x and y are not both in N_1, N_2, or N_3, the pair xy should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$;*
- *for x and y in N_3, the pair xy could be an edge or a non-edge.*

If the nullity k is achievable by some matrix in $\mathcal{S}(G)$, then

$$k \leq \max\{M_{N_0}(H) : H \in \eta_{i-j}(G)\} \leq \max\{M(H) : H \in \eta_{i-j}(G)\},$$

where $M_{N_0}(H)$ means the maximum nullity among all matrices A in $\mathcal{S}(H)$ and the zz-entry of A is nonzero for all $z \in N_0$.

Proof. The 2×2 matrix α in the proof of Theorem 36 is now of the form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ and so invertible. Thus the matrix α^{-1} is of the same form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. Furthermore, for $z \in N_0$, we know the zz-entry of $a\alpha^{-1}a$ is nonzero since all considered matrices are symmetric. $\square$

Example 41. Let G be a multi-partite with three more parts and three more vertices in each part. By some discussion we know that $Z(G) = n-2$, where n is the number of vertices of G. Furthermore, each vertex is a zero-vertex in $\mathcal{I}_{n-2}(G)$. To apply Corollary 40, we may pick arbitrary i and j such that $ij \in E(G)$ and get that N_0 contains the vertices not in parts containing i or j. Therefore N_0 is not empty. Since all graphs in η_{i-j} are graphs with $n-2$ vertices, we know that $M_{N_0}(H) < n-2$ for all H in η_{i-j} since N_0 is not empty. If the nullity $n-2$ is achievable, then we get the contradiction $n-2 < n-2$ by Corollary 40. Thus we know that $M(G) < n-2$.

9 Summary about upper bounds of $M(G)$

In this section, we discuss some other bounds for the maximum nullity of a simple graph and the minimum rank of a pattern matrix. And these bounds could be computed by the zero forcing number on bipartite.

In survey [10], there is an upper bound for the maximum nullity of a simple graph.

Definition 13. A *looped graph* is a graph that allows loops. A vertex x is a neighbor of itself if and only if there is a loop on it. The *zero forcing process on a looped graph* $\widehat{G}$ is the coloring process with the following rules:

- Each vertex of $\widehat{G}$ is either black or white initially.
- If y is the only white neighbor of x, then change the color of y to black.

A set $F \subseteq V(\widehat{G})$ is called a *zero forcing set* if with the initial condition that the vertices in F are black and the remaining vertices are white, each vertex of G could be forced into black by zero forcing process. And the *zero forcing number* $Z(\widehat{G})$ is the minimum size of a zero forcing set.

Definition 14. Let G be a simple graph. The *enhanced zero forcing number* $\widehat{Z}(G)$ is the maximum of $Z(\widehat{G})$ over all looped graph $\widehat{G}$ obtained from G by adding loops on vertices of G.

Theorem 42 ([10]). *For any graph G,*

$$M(G) \leq \widehat{Z}(G) \leq Z(G).$$

Thus we get two bounds for $M(G)$ sharper than $Z(G)$, called $\widehat{Z}(G)$ and $\widetilde{Z}(G)$. But actually they are the same by the following theorem.

Theorem 43. *Let G be a simple graph with n vertex and $I \subseteq [n] = V(G)$ is an index set. Denote $\widehat{G}_I$ to be the looped graph obtained from G by adding loops on vertices in I and $\widetilde{G}_I$ be the graph defined in Section 7. Then*

$$Z(\widehat{G}_I) = Z_Y(\widetilde{G}_I) - n,$$

where the zero forcing processes use the rules on looped graphs and rules on graphs with banned edges respectly.

Proof. We use the notation in the construction of $\widetilde{G}_I$. With the same ordering of vertices, we say that $\{v_i\}_{i\in[I]}$ is the set of vertices of $\widehat{G}_I$. If $i \in I$, then v_i can force itself and b_i can force a_i. If $i \notin I$, then v_i can force others without being black and b_i can force others even if a_i is not black. Observe that the fact that v_i can forces v_j if and only if b_i forces a_j. Two zero forcing processes do the corresponding coloring steps on $\widehat{G}_I$ and $\widetilde{G}_I$. Hence $\{v_i\}_{i\in J}$ is a zero forcing set in $\widehat{G}_I$ if and only if $\{a_i\}_{i\in J}\cup Y$ is a zero forcing set in $\widetilde{G}_I$, where $J \subseteq [n]$ is some index set. □

On the other hand, we called a pattern matrix Q to be a zero-nonzero pattern matrix if the entries of Q contains no u. Paper [3] provides a bound for zero-nonzero pattern matrix. A *t-triangle* of Q is a $t \times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero. The *triangular number* of pattern Q, denote by $\mathrm{tri}(Q)$, is the maximum size of triangle in Q. It was shown that

$$\mathrm{mr}(Q) \geq \mathrm{tri}(Q).$$

And we may observe that the triangle number and the rank of Q are the same by the following theorem.

Theorem 44. *Let Q be a zero-nonzero pattern matrix. Then*

$$\mathrm{tri}(Q) = \mathrm{rank}(Q).$$

Proof. We prove this theorem by showing that

$$\mathrm{tri}(Q) + Z_Y(G) = 2n,$$

where G is the bipartite given be Q in Section 6.

Again, use the notation in the construction of G. If now $k = \mathrm{tri}(Q)$. We may assume, by permutation, Q has a k-triangle in the first k rows and the first k columns. Now set

$$F = \{a_i\}_{i>k} \cup Y.$$

Thus we know that the neighbors of b_i, for $i \leq k$, must be contained in $\{a_j\}_{j\leq i} \cup \{a_j\}_{j>k}$. This means b_1 can force a_1. And b_2 can force a_2 after a_1 becomes black. Inductively, we can force a_k finally. Hence $F \supseteq Y$ is a zero forcing set with support Y. This means that

$$Z_Y(G) \leq |F| = 2n - k = 2n - \mathrm{tri}(Q).$$

Conversely, if $Z_Y(G) = 2n - k$, we can find a set $F \supseteq Y$ with size $2n-k$ such that F is a zero forcing set. We may relabel those vertices such that

$$F = \{a_i\}_{i>k} \cup Y$$

and b_i forces a_i, for $i \le k$, in the chronological list. Since b_1 can force a_1 initially, we know the neighbors of b_1 contains only some elements in F and a_1. This means that the first column of Q has its j-th entry to be zero for all $2 \le j \le k$ and its first entry to be nonzero. Similarly, the neighbors of b_i contains only some elements in $\{a_j\}_{j\le i-1} \cup \{a_j\}_{j>k}$ and a_i. So the i-th column has its j-th entry to be zero for all $i+1 \le j \le k$ and its i-th entry to be nonzero. Thus the first k columns and the first k rows contain a k-triangle. This means that

$$\operatorname{tri}(Q) \ge k = 2n - Z_Y(G).$$

$\square$

Also, now let Q be an $m \times n$ zero-nonzero pattern matrix with the property that each row and each column contain at least one nonzero element $*$. Denote Q' to be the $(m+n)\times(m+n)$ zero-nonzero pattern matrix

$$Q' = \begin{pmatrix} * & Q \\ Q^\top & * \end{pmatrix},$$

were $*$ is the pattern with all entries to be $*$ of appropriate size, and H to be the corresponding simple graph given by Q'.[5] It was shown in paper [3] that

$$\operatorname{mr}(Q) = \operatorname{mr}(Q') = \operatorname{mr}(H)$$

when the considered matriices are over the field of real numbers. So $m+n-Z(H)$ is also a lower bound for $\operatorname{mr}(Q)$. Again, the following theorem shows that this value is exactly the rank of Q.

Theorem 45. *Let Q be a $m \times n$ zero-nonzero pattern matrix with the property that each row and each column contain at least one nonzero element $*$. Then*

$$Z(H) = Z_Y(G),$$

where H is the graph given before this theorem and G is the bipartite constructed from Q in Section 6.

[5]The diagonal entries of Q' are $*$. But this will not disturb the structure of the simple graph H.

Proof. The two graph H and G has the same number of vertices. And the only difference between G and H is the two independent sets in the bipartite G are now two cliques in H. So we may label the corresponding vertices in G and H the same name. So we have

$$V(G) = V(H) = X \cup Y,$$

where

$$X = \{a_1, a_2, \ldots, a_m\},\ Y = \{b_1, b_2, \ldots, b_n\}.$$

For a given optimal zero forcing set $F \supseteq Y$ of G, observe that in the chronological list given by F, we always use b_i to force a_j for some i and j. Hence they will not use the edges in X and Y. Also, the number of neighbors of b_i in X does not increase. So F is also a zero forcing set of H. This implies

$$Z(H) \le Z_Y(G).$$

Now suppose F is an optimal zero forcing set of H. First if F contains Y, then F will also be a zero forcing set with support Y of G since each vertex in $F \cap X$ cannot force others until it remains only one white vertex. In this case, we have done since $Z(H) \ge Z_Y(G)$. Similar arguments for the case $F \supseteq X$ implies that

$$Z(H) \ge Z_X(G) = Z_Y(G)$$

by Theorem 22. Now we may assume that $|X - F|$ and $|Y - F|$ are not zero. If both of $|X - F|$ and $|Y - F|$ are greater than or equal to 2, then every vertex has at least two white neighbors. So this is impossible. Hence we assume first that $|X - F| > |Y - F| = 1$ and $Y - F = \{y\}$. Since $|X - F| > 1$, those vertices in X cannot force others. To begin the zero forcing process, there must be some vertex, denoted by z, in Y such that it forces y. This means that all the neighbors in X of z are elements of F. By our hypothesis, z has at least one neighbor in X. Denote it by x. Thus we have the set $F - x + y$ is a zero forcing set with the same size of F since z will force x at the beginning. Thus we get

$$Z(H) = |F| = |F - x + y| \ge Z_Y(G)$$

since $F - y + x$ is again a set containing Y. Similar argument could be applied to the case $|Y - F| > |X - F| = 1$ and the case $|X - F| = |Y - F| = 1$. In all cases, we will have

$$Z(H) \ge Z_Y(G).$$

So this completes the proof. □

It is a little bit depressed that we spend some time to proof a theorem who tells us that we get no new bound by transforming a minimum rank problem of a pattern matrix to a minimum rank problem of a simple graph. However, the theorem still tell us some rule of possible zero forcing set of H.

Corollary 46. *Let H and G be the graphs in Theorem 45. All the optimal zero forcing set of H must be of one of the following forms.*

1. *F, for some optimal zero forcing set F of G with support X or Y.*
2. *$F - x + y$, for some optimal zero forcing set F of G with support X and for some vertices $x \in X$ and $y \in Y$.*
3. *$F - y + x$, for some optimal zero forcing set F of G with support Y and for some vertices $x \in X$ and $y \in Y$.*

Furthermore, in the second case, there always exists one common neighbor $z \in X$ of x and y such that all the neighbors of z are elements in F except y. In the third case, there always exists one common neighbor $z \in Y$ of x and y such that all the neighbors of z are elements in F except x.

Proof. This is the result in the proof of Theorem 45. □

This corollary could help us find a counterexample given in the next section.

10 A counterexample to a problem on edge spread

Similar to the zero spread of a vertex, we denote the *zero spread of an edge e*, or the *edge spread* without confusion, to be the value $z_e(G) = Z(G) - Z(G - e)$. Theorem 2.21 in paper [8] said that if $z_e(G) = -1$, then for every optimal zero forcing chain set of G, e is an edge in a chain. Also, in Question 2.22, the author asked that whether the converse of Theorem 2.21 is true. Unfortunately, the following is a counterexample saying that e is always used in any optimal chain, but the zero spread is 0 but not -1.

Example 47. A *turtle graph T* is defined by the graph in Figure 16.

We may construct a graph G from T. The vertex set of G is

$$V(G) = X \cup Y,$$

where

$$X = \{a_1, a_2, \ldots, a_{14}\},\ Y = \{b_1, b_2, \ldots, b_{14}\}.$$

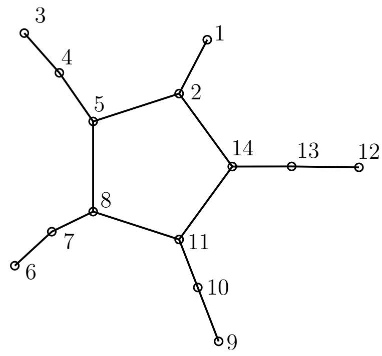

Figure 16: The turtle graph T.

The edge set of G is

$$E(G) = E_1 \cup E_2,$$

where

$$E_1 = \{a_i a_j : i \neq j\} \cup \{b_i b_j : i \neq j\}, \ E_2 = \{a_i b_j : ij \in E(T) \text{ or } i = j\}.$$

Specify the edge e to be $a_1 b_1$. The later proof will tell the fact that all optimal chronological list of Z will use e to obtain a chain but we have

$$Z(G) - Z(G - e) = 0 \neq -1.$$

Proof. To see the behavior of zero forcing set of G, we need the auxiliary graph $G' = (X \cup Y, E_2)$.

First we claim that $Z_Y(G') = 16$ and the only possible form of optimal zero forcing set of G' is

$$F_0 = Y \cup \{u, v\}$$

or all its automorphism types, where u could be a_3 or a_4 and v could be a_6 or a_7. It is easy to see that F_0 is a zero forcing set with size 16. So we have $Z_Y(G') \leq 16$. If now $Z_Y(G') \leq 15$, there is only one addtional vertex in X could be chosen. This is impossible. So we know F_0 is an optimal zero forcing set. Next we observe that if a_3 or a_4 are chosen, then a_3, a_4, a_5 will become black by zero forcing process. So it's impossible to choose two vertices in the set $\{a_3, a_4, a_5\}$. Similarly, we know that the two chosen vertices should come from two of the following sets

$$V_1 = \{a_1, a_2\}, \ V_2 = \{a_3, a_4, a_5\}, \ V_3 = \{a_6, a_7, a_8\},$$

$$V_4 = \{a_9, a_{10}, a_{11}\}, \ V_5 = \{a_{12}, a_{13}, a_{14}\}.$$

Also, the two chosen sets must be consecutive otherwise the zero forcing process would stop after the two sets become totally black. So we may try the left possibilities,

$$(V_1, V_2),\ (V_2, V_3),\ (V_3, V_4).$$

However, the only possibility could only be (V_2, V_3). And we can find out that a_5 and a_8 cannot be chosen. So the set F_0 is the only possibility.

One the other hand, by Corollary 46, we know the optimal zero forcing set of G must be of one of the following types.

- F_0 or its automorphism types.
- $\{a_3, a_4, p\} \cup (Y - y_1)$ or $\{a_6, a_7, q\} \cup (Y - y_2)$ or its automorphism types, where p could be a_6 or a_7, q could be a_3 or a_4, and y_1, y_2 are arbitrarily vertices in Y but $y_1 \neq b_3$ and $y_2 \neq b_6$.

It is a tedious job. But we may check that every optimal zero forcing set of G must pass through the edge $e = a_1 b_1$. Thus we have finished one part of the argument.

Finally, we may observe that

$$F_1 = \{a_6, a_9\} \cup Y$$

is a zero forcing set of $G - e$ with size 16. Thus we know that $Z(G - e) \leq 16$. With the help of Theorem 2.23 in the same paper [8], we get that $Z(G - e)$ could only be 16 and

$$z_e(G) = 16 - 16 = 0 \neq -1.$$

Thus we finish the argument. □

11 Further work

For a given graph, it is still hard to find its maximum nullity unless the lower and the upper bounds meet. In general we just get some possible integers instead of an unique value. If one could derive the reduction formula for k-seperate graphs, which means several graphs with number of common vertices less than k, it would get the exact value of the minimum rank.

On the other hand, the exhaustive zero forcing number is usually too complex to compute. If one could get the reduction formula of vertex-sum for the exhaustive zero forcing number, it would be a nice way to get the upper bound of the minimum rank. Taking another point of view of the exhaustive zero forcing number, in the construction of it and even the zero forcing number, we never used the condition "symmetry". It is reasonable to believe that there is one another parameter between $M(G)$ and $\widetilde{Z}(G)$. And it may be sharp for cactus graphs.

For the pattern

$$Q = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix},$$

we know the minimum rank of Q is 2. However, if we denote $\operatorname{mrs}(Q)$ to be the minimum rank among matrices who are symmetric and of the pattern Q, we will find out that $\operatorname{mrs}(Q)$ is 3 but not 2. This illustrates the importance of symmetry. It is hard to believe a parameter constructed without considering the symmetry will always sharp.

Furthermore, if Q is the pattern of a graph G and Q_I denote the pattern whose zero-nonzero pattern of diagonal entries are given by I, then we have

$$\operatorname{mr}(G) = \operatorname{mrs}(Q) = \min_{I \subset V(G)} \operatorname{mrs}(Q_I).$$

Therefore, the zero-nonzero symmetric minimum rank problem plays an important role if we want to find the minimum rank of a graph.

Finally, the proof of $M(C_n) = 2$, where C_n is a circle of n vertices, in [13] is an idea to test that of which pattern a vector could be in the range space of matrices of given pattern. The idea might be generalized by the concept of independence of sign vectors.

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Appendix: List of computer programs

Sage is a open-source software. In [7], authors gave a program in Sage to compute the bounds of minimum ranks. In this article, some zero forcing numbers were compute by the program. And here we provide a program for computing zero forcing set and number banned by some edges with given support. It was called generalized zero forcing set (gfzs) and number (gZ) in the program instruction. Also, the program can compute the exhaustive zero forcing number (EZ). Since the program contains few concepts of algorithm, the computation of EZ still costs lots of time.

The following is the program listing. One may save it in EZ.sage and type in "load EZ.sage" in the sage software.

```
def gzerosgame(g,F=[],B=[]):
    """
    Return the derived set for a given graph g with set of banned↩
        ↪ edges B and a initial set of vertices. The derived set ↩
        ↪is given by doing generalized zero forcing process. That ↩
        ↪is, if y is the only white neighbor of x and xy is not ↩
        ↪banned, then x could force y into black.

    Input:
        g: a simple graph
        F: a list of vertices of g
        B: a list of tuples representing banned edges of g

    Output:
        A set of black vertices when zero forcing process ↩
            ↪stops.

    Examples:
        sage: gzerosgame(graphs.PathGraph(5),[0])
        set([0, 1, 2, 3, 4])
        sage: gzerosgame(graphs.PathGraph(5),[0],[(1,2)])
        set([0, 1])
    """
    S=set(F) # suspicuous vertices
    Black_vertices=set(F) # current black vertices
    again=1 # iterate again or not
    while again==1:
        again=0
        for x in S:
```

```
            N=set(g.neighbors(x))
            D=N.difference(Black_vertices) # set of white neighbors
            if len(D)==1:
                for v in D:
                    y=v # the only white neighbor
                if (((x,y) in B)==False) and (((y,x) in B)==False):
                    again=1
                    S.remove(x)
                    S.add(y)
                    Black_vertices.add(y)
                break
    return(Black_vertices)

def gZ_leq(graph, support=[], bannedset=[],i=None):
    """
    For a given graph with support and banned set, if there is a zero forcing set of size i then return it; otherwise return False.

    Input:
        graph: a simple graph
        support: a list of vertices of g
        bannedset: a list of tuples representing banned edges of graph
        i: an integer, the function check gZ <= i or not

    Output:
        if F is a zero forcing set of size i and support is a subset of F, then return F
        False otherwise

    Examples:
        sage: gZ_leq(graphs.PathGraph(5),[],[],1)
        set([0])
        sage: gZ_leq(graphs.PathGraph(5),[],[(0,1)],1)
        False
    """
    if i < len(support):
#       print 'i cannot less than the cardinality of support'
        return False
```

```
    j=i-len(support) # additional number of black vertices
    VX=graph.vertices()
    order=graph.order()
    for y in support:
        VX.remove(y)
    # VX is the vertices outside support now
    for subset in Subsets(VX,j):
        test_set=set(support).union(subset) # the set is tested to be a zero forcing set
        outcome=gzerosgame(graph, test_set, bannedset)
        if len(outcome)==order:
            return test_set
    return False

def find_gzfs(graph, support=[], bannedset=[], upper_bound=None, lower_bound=None):
    """
    For a given graph with support and banned set, return the an optimal generalized zero forcing set. If upper_bound is less than the generalized zero forcing number then return ['wrong']. If lower_bound is greater than the generalized zero forcing number then the return value will not be correct

    Input:
        graph: a simple graph
        support: a list of vertices of g
        bannedset: a list of tuples representing banned edges of graph
        upper_bound: an integer supposed to be an upper bound of gZ.
        lower_bound: an integer supposed to be a lower bound of gZ. The two bounds may shorten the computation time. But one may leave it as default value if one is not sure.

    Output:
        if F is an optimal zero forcing set of size i then return F. If upper_bound is less than the general zero forcing number then return ['wrong'].

    Examples:
```

```
            sage: find_gzfs(graphs.PathGraph(5))
            set([0])
            sage: find_gzfs(graphs.PathGraph(5),[1],[(3,2)])
            set([0, 1, 3])
        """

        VX=graph.vertices()
        order=graph.order()
        s=len(support)
        for y in support:
            VX.remove(y)
        # VX is the vertices outside support now
        if upper_bound==None:
            upper_bound=order # the default upper bound
        if lower_bound==None:
            lower_bound=len(VX) # temporary lower bound
            for v in VX:
                N=set(graph.neighbors(v))
                D=N.difference(support)
                lower_bound=min([lower_bound,len(D)])
            for v in support:
                N=set(graph.neighbors(v))
                D=N.difference(support)
                lower_bound=min([lower_bound,len(D)-1])
            lower_bound=lower_bound+s # the default lower bound
        i=upper_bound
        find=1 # does sage find a zero forcing set of size i
        outcome=['wrong'] # default outcome
        while i>=lower_bound and find==1:
            find=0
            leq=gZ_leq(graph, support, bannedset,i) # check gZ <= i or not
            if leq!=False:
                outcome=leq
                find=1
                i=i-1
        return outcome

def find_gZ(graph, support=[], bannedset=[], upper_bound=None, lower_bound=None):
        """
        For a given graph with support and banned set, return the
```

```
        zero. upper_bound and lower_bound could be left as default value if one is not sure.

        Input:
            graph: a simple graph
            support: a list of vertices of g
            bannedset: a list of tuples representing banned edges of graph
            upper_bound: an integer supposed to be an upper bound of gZ.
            lower_bound: an integer supposed to be a lower bound of gZ. The two bounds may shorten the computation time. But one may leave it as default value if one is not sure.

        Output:
            the generalized zero forcing number

        Examples:
            sage: find_gZ(graphs.PathGraph(5))
            1
            sage: find_gZ(graphs.PathGraph(5),[1],[(3,2)])
            3
        """
        return len(find_gzfs(graph, support, bannedset, upper_bound, lower_bound))

def X(g):
        """
        For a given graph g, return the verices set X of a part of the bipartite used to compute the exhaustive zero forcing number.

        Input:
            g: a simple graph

        Output:
            a list of tuples ('a',i) for all vertices i of g

        Examples:
            sage: X(graphs.PathGraph(5))
            [('a', 0), ('a', 1), ('a', 2), ('a', 3), ('a', 4)]
```

```
    """
    return [('a',i) for i in g.vertices()]

def Y(g):
    """
    For a given graph g, return the verices set Y of the other part of the bipartite used to compute the exhaustive zero forcing number.

    Input:
        g: a simple graph

    Output:
        a list of tuples ('b',i) for all vertices i of g

    Examples:
        sage: Y(graphs.PathGraph(5))
        [('b', 0), ('b', 1), ('b', 2), ('b', 3), ('b', 4)]
    """
    return [('b',i) for i in g.vertices()]

def tilde_bipartite(g,I=[]):
    """
    For a given graph g and an index set I, return the bipartite graph \widetilde{G}_I used to compute the exhaustive zero forcing number.

    Input:
        g: a simple graph
        I: a list of vertices of g

    Output:
        the bipartite graph \widetilde{G}_I

    Examples:
        sage: h=tilde_bipartite(graphs.PathGraph(5),[1])
        sage: h.vertices()
        [('a', 0), ('a', 1), ('a', 2), ('a', 3), ('a', 4), ('b', 0), ('b', 1), ('b', 2), ('b', 3), ('b', 4)]
        sage: h.edges()
        [(('a', 0), ('b', 1), None), (('a', 1), ('b', 0), None), (('a', 1), ('b', 1), None), (('a', 1), ('b
```

```
        ↪', 2), None), (('a', 2), ('b', 1), None), (('a', ↩
        ↪2), ('b', 3), None), (('a', 3), ('b', 2), None), ↩
        ↪(('a', 3), ('b', 4), None), (('a', 4), ('b', 3), ↩
        ↪None)]
    """
    E0=[(('a',i), ('b',i)) for i in I] # edges given by I
    E1=[] # invariant edges
    for i in g.vertices():
        for j in g.neighbors(i):
            E1.append((('a',i),('b',j)))
    h=Graph()
    h.add_vertices(X(g))
    h.add_vertices(Y(g))
    h.add_edges(E0)
    h.add_edges(E1) # h=(X union Y, E0 union E1)
    return h

def find_EZ(g,bound=None):
    """
    For a given graph g, return the exhaustive zero forcing ↩
      ↪number of g. A given bound may shorten the computation.

    Input:
        g: a simple graph
        bound: a integer as an upper bound. It could be left ↩
          ↪as default value if one is not sure.

    Output:
        the exhaustive zero forcing number (EZ) of g

    Examples:
        sage: find_EZ(graphs.PathGraph(5))
        1
        sage: h=graphs.CycleGraph(5)
        sage: h.add_vertices([5,6,7,8,9])
        sage: h.add_edges([(0,5),(1,6),(2,7),(3,8),(4,9)])
        sage: find_EZ(h) # the EZ of a 5-sun
        2
    """
    order=g.order()
    Z=find_gZ(g) # without support and banned set, the value is ↩
      ↪the original zero forcing number
```

```
if bound==None:
        bound=Z # default upper bound
gZ_bound=bound+order
V=set(g.vertices())
e=-1 # temporary output
for I in Subsets(V):
        leq=gZ_leq(tilde_bipartite(g,I),Y(g),[],e) # this avoid abundant computation
        if leq==False:
                e=find_gZ(tilde_bipartite(g,I),Y(g),[],gZ_bound,e+1)
                # in this case, we already know e+1-order<=gZ-order<=bound and so e+1<=gZ<=gZ_bound
        if e==gZ_bound:
                break
return e-order # EZ=max-order
```