Variants of zero forcing and their applications to the minimum rank problem

by

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DEDICATION

To my family in Taiwan

and my friends in Tortuga.

—Jephian

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ABSTRACT

The minimum rank problem refers to finding the smallest possible rank, or equivalently the largest possible nullity, among matrices under certain restrictions. These restrictions can be the zero-nonzero pattern, conditions on the inertia, or other properties of a matrix. Zero forcing is a powerful technique for controlling the nullity and plays a significant role in the minimum rank problem. This thesis introduces several zero forcing parameters and their applications on the minimum rank problem.

Zero-nonzero patterns can be described by graphs: The edges (including the loops) represent the nonzero entries, while the non-edges correspond to the zero entries. For simple graphs, where no loops are allowed, the diagonal entries can be any real numbers. The maximum nullity of a graph is the maximum nullity among symmetric matrices with the pattern described by the graph. In Chapter 2, the odd cycle zero forcing number $Z_{oc}(\mathfrak{G})$ and the enhanced odd cycle zero forcing number $\widehat{Z}_{oc}(G)$ are introduced as bounds for the maximum nullities of loop graphs $\mathfrak{G}$ and simple graphs G, respectively. Also, a relation between loop graphs and simple graphs through graph blowups is developed.

The Colin de Verdière type parameter $\xi(G)$ is defined as the maximum nullity of real symmetric matrices A with the pattern described by G and with the Strong Arnold Property (SAP), which means $X = O$ is the only symmetric matrix that satisfies $A \circ X = I \circ X = AX = O$ (here $\circ$ is the entrywise product). Chapter 3 introduces zero forcing parameters $Z_{\mathrm{SAP}}(G)$ and $Z_{\mathrm{vc}}(G)$; we show that $Z_{\mathrm{SAP}}(G) = 0$ implies every symmetric matrix with the pattern described by G has the SAP and that the inequality $M(G) - Z_{\mathrm{vc}}(G) \leq \xi(G)$ holds for every graph G. Also, the values of $\xi(G)$ are computed for all graphs up to 7 vertices, establishing $\xi(G) = \lfloor Z \rfloor(G)$ for these graphs.

CHAPTER 1. INTRODUCTION

The minimum rank problem is to determine the smallest rank or the largest nullity over a family of matrices under certain restrictions. These restrictions can be the zero-nonzero pattern, conditions on the inertia, or other properties of a matrix. Since the sum of the rank and the nullity is the dimension of a matrix, finding the minimum rank and finding the maximum nullity are equivalent. A tool that frequently appears in many aspects of the minimum rank problem is zero forcing. Zero forcing aims to control the nullity of a (homogeneous) linear system through the following mechanics. At the beginning, we pick some variables and mark them as blue, while others are marked as white. Assume all the blue variables equal to zero and the values of white variables are unknown; if for some equation, all variables with nonzero coefficients are blue (zero) except for one, then this remaining variable turns blue (zero), because the given information forces it to be zero. If starting with a blue/white coloring can eventually make all variables blue, then the number of initial blue variables is an upper bound of the nullity of the linear system. This thesis introduces several new zero forcing parameters and studies their applications to the minimum rank problem.

A classical minimum rank problem concerns the maximum nullity of a graph and has been studied extensively; see [9,10] for surveys. Let G be a simple graph on n vertices. Its associated family $\mathcal{S}(G)$ of matrices collects those $n \times n$ real symmetric matrices whose off-diagonal i,j-entry is nonzero whenever i and j are adjacent in G. The *maximum nullity* of G is the largest nullity of matrices in $\mathcal{S}(G)$ and is denoted by $M(G)$. The study of $M(G)$ is motivated by the inverse eigenvalue problem of a graph, which asks the possible spectra of matrices in $\mathcal{S}(G)$; each eigenvalue in the spectrum can have its multiplicity at most $M(G)$. By changing the set of matrices considered, variants of the maximum nullity are defined and have their own applications. The *maximum positive semidefinite nullity* $M_+(G)$ is the largest nullity for positive semidefinite matrices in $\mathcal{S}(G)$, and it computes the minimum dimension of a linear space where an orthogonal representation of G exists [14]. The Haemers rank and the Colin de Verdière parameter are defined in a similar flavor; the Haemers rank gives an upper bound for the Shannon capacity [13], while the Colin de Verdière parameter characterizes planar graphs [7]. The maximum nullity can also be defined on loop graphs, where the nonzero/zero pattern on the diagonal is given by the loops.

The *zero forcing number* $Z(G)$ can be defined through a color-change game on vertices of a graph G, where vertices are blue or white; $Z(G)$ is defined as the minimum number of blue vertices required initially so that all vertices will turn blue at the end by repeated applications of the following color-change rule: If x is a blue vertex and y is the only white neighbor of x in G, then y turns blue. The zero forcing number $Z(G)$ was introduced as an upper bound of the maximum nullity $M(G)$ [1]. Zero forcing number was also introduced by physicists independently for the study of quantum control [5]. Few years later, variants of zero forcing was found to be related to the fast-mixed search in computer science [11] and the cops-and-robber game in graph theory [2]. Zero forcing can also be used for designing logic circuits [6]. Many variants of the maximum nullity are bounded above by variants of the zero forcing number; see [2] for the related parameters and their relations.

The Colin de Verdière type parameters are the maximum nullities over certain matrices with the Strong Arnold Property; they draw the attention of graph theorists because of the minor-monotonicity of these parameters in addition to other important properties. A real symmetric matrix A is said to have the *Strong Arnold Property* (SAP) if $X = O$ is the only symmetric matrix that satisfies $A \circ X = I \circ X = AX = O$, where $\circ$ is the entrywise product. The *Colin de Verdière parameter* $\mu(G)$ is the maximum nullity over matrices in $\mathcal{S}(G)$ with non-positive off-diagonal entries, exactly one negative eigenvalue, and the SAP. The parameter $\mu(G)$ has nice topological properties: A graph G is planar if and only if $\mu(G) \leq 3$, is outer-planar if and only if $\mu(G) \leq 2$, and is a disjoint union of paths if and only if $\mu(G) \leq 1$ [7]. Also, if a graph H is a minor of another graph G, then $\mu(H) \leq \mu(G)$. This property is called the *minor-monotonicity*, and all Colin de Verdière type parameters are minor-monotone [3, 7, 8]. As a result of the graph minor theorem, these parameters can be computed for small values by checking a known (finite) list of forbidden minors.

1.1 Overview

In Chapter 2, we will consider the maximum nullity of simple graphs and that of loop graphs. Upper bounds utilizing zero forcing will be given for both types of graphs. When a simple graph is a blowup of a loop graph, we will establish a relation between their maximum nullities and their zero forcing numbers. Chapter 2 will also consider the maximum nullities $M^F(G)$ over other fields F; when the field is $\mathbb{R}$, we simply write it as $M(G)$.

A simple graph is a graph without loops or multiedges, while a *loop graph* is a graph where each vertex can have at most one loop (but not multiple edges). For a loop graph $\mathfrak{G}$, the associated family $\mathcal{S}(\mathfrak{G})$ of matrices consists of those real symmetric matrices whose i, j-entry is nonzero whenever $\{i, j\}$ is an edge (or a loop) in $\mathfrak{G}$, and the *maximum nullity* $M(\mathfrak{G})$ is the largest nullity over matrices in $\mathcal{S}(\mathfrak{G})$. Therefore, the loops control the diagonal entries. The simple graphs and loop graphs are bridged by the notion of loop configurations. For a simple graph G, a *loop configuration* of G is a loop graph $\mathfrak{G}$ obtained from G by designating each vertex as having or not having a loop. By definition, $M(G) = \max_{\mathfrak{G}} M(\mathfrak{G})$, where the maximum is over all loop configurations $\mathfrak{G}$ of G.

The *zero forcing number* $Z(\mathfrak{G})$ of a loop graph $\mathfrak{G}$ is the minimum number of blue vertices required so that all vertices can turn blue by repeated applications of the following color-change rule: If y is the only white neighbor of x, then y turns blue. Here x is considered as a neighbor of itself if and only if there is a loop on it. Notice that in this color-change rule, x is not required to be blue to make a force. It is known [15] that $M(\mathfrak{G}) \leq Z(\mathfrak{G})$ for every loop graph $\mathfrak{G}$. Through the loop configurations, one can easily shift an upper bound on loop graphs to an upper bound on simple graphs. That is, $\widehat{Z}(G) = \max_{\mathfrak{G}} Z(\mathfrak{G})$, where the maximum is over all loop configurations $\mathfrak{G}$ of G, is an upper bound of $M(G)$. Indeed, it is known [2] that $M(G) \leq \widehat{Z}(G) \leq Z(G)$, and each of the inequalities can be strict.

A family of loop graphs $\mathfrak{G}$ such that $M(\mathfrak{G}) < Z(\mathfrak{G})$ is the loopless odd cycles. Let C_n denote the simple graph of a cycle on n vertices. The *loopless odd cycle* $\mathfrak{C}^0_{2k+1}$ is a loop configuration of C_{2k+1} with no loop on each vertex. It is known [4] that $M(\mathfrak{C}^0_{2k+1}) = 0$ and $Z(\mathfrak{C}^0_{2k+1}) = 1$ for all $k \geq 1$. This fact suggests a way to design a new zero forcing number by modifying the color-change rule with one more statement, "whenever a loopless odd cycle appears as a component of the subgraph induced on the current white vertices, make all vertices on it blue." In Section 2.2, we will define the *odd cycle zero forcing number* $Z_{oc}(\mathfrak{G})$ and show that $M(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$. For many graphs with $M(\mathfrak{G}) < Z(\mathfrak{G})$, now we have $M(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$.

In Section 2.3, we will define the *enhanced odd cycle zero forcing number* as $\widehat{Z}_{oc}(G) = \max_{\mathfrak{G}} Z_{oc}(\mathfrak{G})$, where the maximum is over all loop configurations $\mathfrak{G}$ of G, and show that $M(G) \leq \widehat{Z}_{oc}(G) \leq \widehat{Z}(G) \leq Z(G)$, providing a new upper bound for $M(G)$. Corollary 2.4.9 and Proposition 2.6.1 provide examples showing that $\widehat{Z}(G) - \widehat{Z}_{oc}(G)$ and $\widehat{Z}_{oc}(G) - M(G)$ can be arbitrarily large.

Graph blowup is a transformation from a loop graph to a simple graph and was used for the characterization of the minimum ranks over finite fields [12]. In Section 2.4, we will define graph blowups and show that $M(H) = \widehat{Z}_{oc}(H)$ if $M(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$, provided that H is a "large" blowup of $\mathfrak{G}$; moreover, the value of $M(H) = \widehat{Z}_{oc}(H)$ can be obtained by the value of $M(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$. In Section 2.5, we will consider the graph complement conjecture for $M(G)$ and show that this conjecture is true for most graph blowups.

Chapter 3 will consider the Strong Arnold Property. The SAP, a property of a matrix, has been used to define the Colin de Verdière type parameters ξ, μ, and ν. The parameter $\xi(G)$ is the largest nullity of matrices in $\mathcal{S}(G)$ with the SAP. Therefore, $\xi(G) \leq M(G)$ and the only difference between their definitions is the SAP. In Section 3.2, we will introduce a new parameter $Z_{\text{SAP}}(G)$ and show that if $Z_{\text{SAP}}(G) = 0$ then every matrix in $\mathcal{S}(G)$ has the SAP, implying $\xi(G) = M(G)$. Based on the computational results in Table 3.1, graphs with this property do not seem to be rare.

In Section 3.3, we will introduce another parameter $Z_{\text{vc}}(G)$ and show that $M(G) - \xi(G) \leq Z_{\text{vc}}(G)$ for every graph G. With the help of $Z_{\text{SAP}}(G)$, $Z_{\text{vc}}(G)$, and some existing theorems, Section 3.4 will provide a way to compute the values of $\xi(G)$ for graphs G up to 7 vertices and show that for such graphs $\xi(G) = \lfloor Z \rfloor(G)$, where $\lfloor Z \rfloor(G)$ is the *minor monotone floor* of the zero forcing number introduced in [2] and defined in Section 3.4.

Variants of $Z_{\text{SAP}}(G)$ and $Z_{\text{vc}}(G)$ are also introduced in Chapter 3 and their relations are illustrated in Figure 3.1.

1.2 Organization of the thesis

This thesis is a collection of papers published in journals. Chapter 1 gives a general overview of the minimum rank problem and its relation with zero forcing. Chapter 2 and 3 are two self-contained papers. Chapter 2 contains the paper "Odd cycle zero forcing parameters and the minimum rank of graph blowups" published in the *Electronic Journal of Linear Algebra* [16]. Chapter 3 contains the paper "Using a new zero forcing process to guarantee the Strong Arnold Property" published in *Linear Algebra and its Applications* [17]. Both papers are individual works of Jephian Chin-Hung Lin under the guidance of his advisors Leslie Hogben and Steve Butler.

Chapter 4 will contains concluding remarks and possible future research directions for the minimum rank problem and zero forcing.

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CHAPTER 2. ODD CYCLE ZERO FORCING PARAMETERS AND THE MINIMUM RANK OF GRAPH BLOWUPS

A paper published in the Electronic Journal of Linear Algebra

Jephian C.-H. Lin

Abstract

The minimum rank problem for a simple graph G and a given field F is to determine the smallest possible rank among symmetric matrices over F whose i,j-entry, $i \neq j$, is nonzero whenever i is adjacent to j, and zero otherwise; the diagonal entries can be any element in F. In contrast, loop graphs $\mathfrak{G}$ go one step further to restrict the diagonal i,i-entries as nonzero whenever i has a loop, and zero otherwise. When $\operatorname{char} F \neq 2$, we introduce the odd cycle zero forcing number and the enhanced odd cycle zero forcing number as bounds for loop graphs and simple graphs respectively. We also build a relation between loop graphs and simple graphs through graph blowups, so that the minimum rank problem of some families of simple graphs can be reduced to that of much smaller loop graphs.

2.1 Introduction

For a given graph, the *minimum rank problem* is to determine the smallest possible rank among a family of matrices associated to the graph. Depending on the types of graphs, the definitions of the associated matrices are different. In this paper, we focus on simple graphs and loop graphs, provide new bounds for both of them, and develop their relation on the minimum rank problem through graph *blowups* (which will be defined in Section 2.4).

A simple graph is a graph without loops or multiedges; a *loop graph* is a graph where each vertex can have at most one loop. Given a field F, the set of associated matrices of a simple graph G is denoted by $\mathcal{S}^F(G)$ and defined as the family of symmetric matrices over F whose i,j-entry, $i \neq j$, is nonzero whenever i is adjacent to j, and zero otherwise; in contrast, the associated matrices $\mathcal{S}^F(\mathfrak{G})$ of a loop graph $\mathfrak{G}$ is the family of symmetric matrices over F whose i,j-entry ($i = j$ is possible) is nonzero whenever i is adjacent to j, and zero otherwise. Note that in a loop graph, i is adjacent to itself if and only if i has a loop. To point out the difference, the diagonal entries can be any element in F for simple graphs; however, for loop graphs, the zero-nonzero pattern on the diagonal is controlled by the loops. A graph without any loops can be considered as a simple graph G or a loop graph $\mathfrak{G}$ without loops, but the definitions for $\mathcal{S}^F(G)$ and $\mathcal{S}^F(\mathfrak{G})$ are different, since $\mathcal{S}^F(G)$ allows free diagonal while $\mathcal{S}^F(\mathfrak{G})$ requires zero diagonal. Therefore, a simple graph is usually denoted as G and a loop graph is denoted as $\mathfrak{G}$.

The *minimum rank* of a given graph, is defined as the smallest possible rank in $\mathcal{S}^F(G)$, or $\mathcal{S}^F(\mathfrak{G})$. For a simple graph G and a loop graph $\mathfrak{G}$, the minimum ranks are written as $\operatorname{mr}^F(G) = \min\{\operatorname{rank}(A) : A \in \mathcal{S}^F(G)\}$ and $\operatorname{mr}^F(\mathfrak{G}) = \min\{\operatorname{rank}(A) : A \in \mathcal{S}^F(\mathfrak{G})\}$ respectively. Equivalently, the problem of finding the minimum rank of a graph can be viewed as finding the *maximum nullity*, which is defined as $M^F(G) = \max\{\operatorname{null}(A) : A \in \mathcal{S}^F(G)\}$ and $M^F(\mathfrak{G}) = \max\{\operatorname{null}(A) : A \in \mathcal{S}^F(\mathfrak{G})\}$. This is because $\operatorname{mr}^F(G) + M^F(G) = |V(G)|$ for any simple graph G, or similarly when G is replaced by any loop graph $\mathfrak{G}$.

The minimum rank problem is a relaxation of the *inverse eigenvalue problem*, and also essentially related to *orthogonal representations* and the *Colin de Verdiére type parameters* (see [10]). For the study of the minimum rank problem, the *zero forcing* *number* Z was introduced in [1], and then [13] extended to each type of graph as a "universal" upper bound for the maximum nullity. That is, $M^F(G) \leq Z(G)$ for any field F and any simple graph G, or when G is replaced by a loop graph $\mathfrak{G}$. Zero forcing parameters will be discussed in Section 2.1.1

In the sense of the maximum nullity and the zero forcing number, the relation between simple graphs and loop graphs is bridged by the *loop configurations*. A loop configuration of a simple graph G is a loop graph $\mathfrak{G}$ obtained from G by designating each vertex as having no loop or one loop. So for a given simple graph G with n vertices, there are 2^n possible loop configurations of G. Through this definition, the maximum nullity of a simple graph can be obtained from the maximum nullities of its loop configurations. That is, $M^F(G) = \max_{\mathfrak{G}} M^F(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of G. Since $M^F(\mathfrak{G}) \leq Z(\mathfrak{G})$ for each of the loop configurations, the *enhanced zero forcing number* $\widehat{Z}(G)$ was introduced in [4] and is defined as $\widehat{Z}(G) = \max_{\mathfrak{G}} Z(\mathfrak{G})$, where the maximum is over all loop configurations $\mathfrak{G}$ of G. In the same paper, it is shown $M^F(G) \leq \widehat{Z}(G) \leq Z(G)$ for any simple graph G and any field F. This suggests that the consideration of loop graphs can improve the upper bound given by $Z(G)$.

For the field of real numbers, it is known [8] that $M^{\mathbb{R}}(G) = Z(G)$ for any simple graph with $|V(G)| \leq 7$, yet this is not the case for loop graphs. For example, let C_n be the cycle on n vertices, as a simple graph. A *loopless odd cycle* $\mathfrak{C}^0_{2k+1}$ is the loop configuration of C_{2k+1} without any loop. For a loopless odd cycle $\mathfrak{C}^0_{2k+1}$, its maximum nullity $M^F(\mathfrak{C}^0_{2k+1}) = 0$ for any field F with characteristic $\operatorname{char} F \neq 2$, but $Z(\mathfrak{C}^0_{2k+1}) = 1$ [7]. That means, even for small loop graphs like $\mathfrak{C}^0_3$, there is a gap.

When $\operatorname{char} F \neq 2$, loopless odd cycles play an important role, and allow us to discover new upper bounds for both loop graphs and simple graphs. In Section 2.2, we define a new parameter called the *odd cycle zero forcing number*, $Z_{oc}(\mathfrak{G})$, for loop graphs $\mathfrak{G}$; meanwhile, Theorem 2.2.8 proves that $M^F(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$, and Corollary 2.2.9 states that $M^{\mathbb{R}}(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ whenever $F = \mathbb{R}$ and $\mathfrak{G}$ is a loop configuration of a complete graph or a cycle, which fixes the gap between $Z(\mathfrak{C}^0_{2k+1})$ and $M^{\mathbb{R}}(\mathfrak{C}^0_{2k+1})$.

Following the same track of the enhanced zero forcing number, when $\operatorname{char} F \neq 2$, the odd cycle zero forcing number for loop graphs also leads to a new bound for simple graphs. In Section 2.3, the *enhanced odd cycle zero forcing number* $\widehat{Z}_{oc}(G)$ for simple graphs is introduced with the property $M^F(G) \leq \widehat{Z}_{oc}(G) \leq \widehat{Z}(G) \leq Z(G)$. Example 2.3.3 shows that $M^{\mathbb{R}}(K_{3,3,3}) = \widehat{Z}_{oc}(K_{3,3,3}) = 6$ and $\widehat{Z}(K_{3,3,3}) = 7$, where $K_{3,3,3}$ is the complete tripartite (simple) graph. Corollary 2.4.9 and Proposition 2.6.1 provide examples showing that $\widehat{Z}(G) - \widehat{Z}_{oc}(G)$ and $\widehat{Z}_{oc}(G) - M^{\mathbb{R}}(G)$ can be arbitrarily large.

Graph blowups are a transformation from a loop graph to a simple graph, and were used for the characterization for minimum rank over finite fields [12]. In Section 2.4, graph blowups are defined, and Theorem 2.4.7 shows that $M^F(H) = \widehat{Z}_{oc}(H)$ if $M^F(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$, provided that H is a "large" blowup of $\mathfrak{G}$. That means the maximum nullity of a graph blowup, which is a simple graph, can be obtained by the maximum nullity of a much smaller loop graph.

In Section 2.5, the *graph complement conjecture* for $M(G)$ is shown to be true for most graph blowups; while the graph complement conjecture for $\widehat{Z}_{oc}(G)$ is true for any simple graph.

2.1.1 Different types of zero forcing numbers

There are several different types of zero forcing numbers, but they all serve as upper bounds for the maximum nullity for different types of graphs. In this section, the zero forcing number $Z(G)$ for simple graphs G and the zero forcing number $Z(\mathfrak{G})$ for loop graphs $\mathfrak{G}$ will be discussed.

The zero forcing number starts by the *zero forcing game*, where vertices are blue or white and different *color-change rules* may apply on different types of graphs. For simple graphs G, the color-change rule is

- if $y \in V(G)$ is the only white neighbor of $x \in V(G)$ and x is blue, then y turns blue;

for loop graphs $\mathfrak{G}$, the color-change rule is

- if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ (where $x = y$ is possible), then y turns blue.

So one of the major differences is for simple graphs, x should be blue first so x can force its neighbor y to turn blue, but for loop graphs, this need not be the case. Also, we emphasize the neighbors of x for loop graphs refers to those vertices which are adjacent to x. So it is possible that x itself is the only white neighbor of x, when there is a loop on x.

On a graph with vertex set V, a subset $B \subseteq V$ is called a *zero forcing set* if setting the vertices of B blue and the others white can make the whole set V change to blue through repeated applications of the corresponding color-change rule. The zero forcing number $Z(G)$, or $Z(\mathfrak{G})$, is defined to be the minimum cardinality of a zero forcing set on a simple graph G, or a loop graph $\mathfrak{G}$, using the appropriate color-change rule.

Suppose $\mathfrak{G}$ is a loop configuration of a simple graph G. Then any zero forcing set of G is a zero forcing set of $\mathfrak{G}$, so $Z(\mathfrak{G}) \leq Z(G)$. This establishes the stated theorem

$$M^F(G) \leq \widehat{Z}(G) \leq Z(G)$$

in [4].

As mentioned, there is a gap between $M^F(\mathfrak{G})$ and $Z(\mathfrak{G})$ when $\mathfrak{G}$ is a loopless odd cycle and $\operatorname{char} F \neq 2$. In fact, this also happens on some loop configurations of complete graphs, when $\mathfrak{C}_3^0$ appears on it. For loop configurations of complete graphs, the maximum nullity can be found in [7] and Proposition 2.1.1 provides the zero forcing number .

Proposition 2.1.1. *Let K_n be the complete (simple) graph on n vertices and $\mathfrak{K}_n^{\ell(s)}$ its* *loop configuration with s loops. Then*

$$M^{\mathbb{R}}(\mathfrak{K}_n^{\ell(s)}) = \begin{cases} n & \text{if } n-s=1=n; \\ n-1 & \text{if } n-s=0 \text{ and } 1\le n; \\ n-2 & \text{if } 1\le n-s\le 2\le n; \\ n-3 & \text{if } 3\le n-s, \end{cases}$$

and

$$Z(\mathfrak{K}_n^{\ell(s)}) = \begin{cases} n & \text{if } n-s=1=n; \\ n-1 & \text{if } n-s=0 \text{ and } 1\le n; \\ n-2 & \text{if } 1\le n-s \text{ and } 2\le n. \end{cases}$$

Proof. Since $M^{\mathbb{R}}(\mathfrak{G})+\mathrm{mr}^{\mathbb{R}}(\mathfrak{G}) = |V(\mathfrak{G})|$ for every loop graph $\mathfrak{G}$, the formula for $M^{\mathbb{R}}(\mathfrak{K}_n^{\ell(s)})$ comes from Proposition 5.5 in [7]. To determine the zero forcing number, two cases are considered. When $n-s \le 2$, the formulas for $Z(\mathfrak{K}_n^{\ell(s)})$ and $M(\mathfrak{K}_n^{\ell(s)})$ agree with each other, so it is enough to find a zero forcing set of cardinality $M(\mathfrak{K}_n^{\ell(s)})$. When $n-s=1=n$, the graph has only one vertex, which makes a zero forcing set. When $n-s=0$ and $1\le n$, any set of $n-1$ vertices can be a zero forcing set. When $1\le n-s\le 2\le n$, any set of $n-2$ vertices with loops forms a zero forcing set.

In the case of $3\le n-s$, by coloring all vertices blue except two vertices without loops, it becomes a zero forcing set. However, if there are 3 white vertices initially, then every vertex will have at least two white neighbors (beside itself), so it cannot be a zero forcing set. As a consequence, $n-2$ is the zero forcing number. □

Proposition 2.1.2. [7] *Let C_n be the (simple) cycle on n vertices and $\mathfrak{C}_n$ one of its loop configurations. Then $M^{\mathbb{R}}(\mathfrak{C}_n) = Z(\mathfrak{C}_n)$ whenever $\mathfrak{C}_n$ is not a loopless odd cycle. For loopless odd cycles $\mathfrak{C}_{2k+1}^0$, $M^{\mathbb{R}}(\mathfrak{C}_{2k+1}^0) = 0$ but $Z(\mathfrak{C}_{2k+1}^0) = 1$.*

Remark 2.1.3. The equality $M^F(\mathfrak{C}_{2k+1}^0) = 0$ holds for any field F with $\operatorname{char} F \ne 2$. This is because a loop graph $\mathfrak{G}$ with a *unique spanning generalized cycle* always has $M(\mathfrak{G}) = 0$ if $\operatorname{char} F \ne 2$ [11] (spanning generalized cycles are called spanning composite cycles in [7]).

This states that every matrix in $\mathcal{S}^F(\mathfrak{C}^0_{2k+1})$ is nonsingular whenever $\operatorname{char} F \neq 2$. On the other hand, $M^F(\mathfrak{C}^0_{2k+1}) = 1$ if $\operatorname{char} F = 2$, because $Z(\mathfrak{C}^0_{2k+1}) = 1$ and the adjacency matrix of $\mathfrak{C}^0_{2k+1}$ over F has determinant 0. Symmetry is also crucial, since the asymmetric matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ follows the zero-nonzero pattern given by $\mathfrak{C}^0_3$ but has nullity 1.

2.2 Odd cycle zero forcing number $Z_{oc}(\mathfrak{G})$

This section exploits Remark 2.1.3 to develop a new upper bound of $M^F(\mathfrak{G})$ for loop graphs $\mathfrak{G}$ when $\operatorname{char} F \neq 2$.

For a loop graph $\mathfrak{G}$ and a subset of vertices $W \subseteq V(\mathfrak{G})$, the *induced subgraph* of $\mathfrak{G}$ on W is the loop graph obtained from $\mathfrak{G}$ by deleting all vertices outside W, which keeps all those edges and loops with their two endpoints in W. The odd cycle zero forcing number is an extension of the conventional zero forcing number by adding one more rule.

Definition 2.2.1. On a given loop graph $\mathfrak{G}$, where vertices are marked blue or white, the *color-change rule for* Z_{oc} (CCR-Z_{oc}) is:

(a) if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ (where $x = y$ is possible), then y turns blue.

(b) if W is the set of white vertices and $\mathfrak{G}[W]$ contains a component $\mathfrak{C}$, as a loopless odd cycle, then all vertices of $\mathfrak{C}$ turn blue.

If starting with $B \subseteq V(\mathfrak{G})$ as initial blue vertices makes the whole set $V(\mathfrak{G})$ change to blue through repeated applications of CCR-Z_{oc}, then B is a *zero forcing set for* Z_{oc} (ZFS-Z_{oc}) on $\mathfrak{G}$. The *odd cycle zero forcing number* is defined as

$$Z_{oc}(\mathfrak{G}) = \min\{|B| : B \text{ is a } \textit{ZFS-}Z_{oc} \text{ on } \mathfrak{G}\}.$$

Remark 2.2.2. Given an initial blue set, no matter what order the rules (a) and (b) are applied, the process always stops at some unique final coloring where neither color-change rule can be used. To see this, suppose at a certain step, W is the set of white vertices and $\mathfrak{C}$ is a loopless odd cycle as a component of $\mathfrak{G}[W]$; also suppose $y \in V(\mathfrak{C})$ is the only white neighbor of x. In this situation, we can apply rule (b), and all vertices in $V(\mathfrak{C})$ turns blue; on the other hand, if we apply rule (a) instead to make y blue, then all vertices in $V(\mathfrak{C})$ will eventually turn blue, since y is a (conventional) zero forcing set of $\mathfrak{C}$. So the order of using rule (a) and rule (b) will not affect the final set of blue vertices. For implementing an algorithm, one can consider rule (b) only when rule (a) no longer applies. (A fast implementation of rule (a) exists but no fast implementation of rule (b) currently exists. This explains our preference for rule (a).)

The following concepts are helpful for understanding this new color-change rule. A *chronological list for Z_{oc}* records how a ZFS-Z_{oc} makes all vertices blue, and is defined as $(X_i \to Y_i)_{i=1}^{s}$, where at the i-th step, if rule (a) is applied, then $X_i = \{x\}$ and $Y_i = \{y\}$, while if rule (b) is applied, then $X_i = Y_i = V(\mathfrak{C})$. Here x, y, and $\mathfrak{C}$ are as those in Definition 2.2.1. A *zero forcing process for Z_{oc}* (ZFP-Z_{oc}) refers to the initial blue set B and its chronological list. Note that a ZFS-Z_{oc} may have different ways of applying CCR-Z_{oc}, so the chronological list for Z_{oc} with a given ZFS-Z_{oc} is not unique. Note that we do not restrict the ZFS-Z_{oc} of a chronological list to be a minimum ZFS-Z_{oc}; when it is minimum, the chronological list and ZFP-Z_{oc} are said to be *optimal*.

For a given chronological list, we can draw a corresponding digraph on $V(\mathfrak{G})$ with arcs indicated by $X_i \to Y_i$. If $X_i = \{x\}$ and $Y_i = \{y\}$, then $x \to y$ is added; if $X_i = Y_i = V(\mathfrak{C})$ for some loopless odd cycle $\mathfrak{C}$, then an odd directed cycle is added, with some circular orientation. With these definitions, each (weakly connected) component of this digraph is called a *maximal chain*.

On a digraph, a sequence of vertices with structure $v_1 \to v_2 \to \cdots \to v_n$ is called a *directed n-path*, where v_1 is called the *tail* and v_n is called the *head* of this directed path;

and a sequence of vertices with the structure $v_1 \to v_2 \to \cdots \to v_n \to v_1$ is called a *directed n-cycle*. On digraphs, a directed 1-cycle or a directed 2-cycle are possible, and they are a vertex with a loop or two vertices with two arcs of both directions.

Example 2.2.3. Let $\mathfrak{G}$ be the loop graph in Figure 2.1 and $B = \{3\}$ be the initial blue set. Then the column on the left is one possible chronological list, while the column on the right is its reversal, which reverses the order of the list and switches the roles of X_i's and Y_i's. The reversal is also a ZFP-Z_{oc}, with the initial blue set $B' = \{6\}$.

$\{2\} \to \{1\}$	$\{10\} \to \{10\}$
$\{1\} \to \{2\}$	$\{7,8,9\} \to \{7,8,9\}$
$\{3\} \to \{4\}$	$\{5\} \to \{4\}$
$\{5\} \to \{6\}$	$\{6\} \to \{5\}$
$\{4\} \to \{5\}$	$\{4\} \to \{3\}$
$\{7,8,9\} \to \{7,8,9\}$	$\{2\} \to \{1\}$
$\{10\} \to \{10\}$	$\{1\} \to \{2\}$

Following this chronological list, its maximal chains are shown in Figure 2.1, including a vertex with a directed 1-cycle on $\{10\}$, a directed 2-cycle on $\{1,2\}$, a directed path on $\{3,4,5,6\}$, and a directed odd cycle on $\{7,8,9\}$ given by rule (b).

Example 2.2.3 shows all possible types of maximal chains. Proposition 2.2.4 and Proposition 2.2.5 develop some general properties for maximal chains.

Proposition 2.2.4. *Let $\mathfrak{G}$ be a loop graph and B a ZFS-Z_{oc} on $\mathfrak{G}$. Let ζ be a ZFP-Z_{oc} with its initial blue set B and Γ the corresponding digraph of ζ. By CCR-Z_{oc}, the following properties hold:*

(1) *for every vertex $x \in B$, the in-degree of x in Γ is* 0*;*

(2) *for every vertex $x \in V(\mathfrak{G}) \setminus B$, the in-degree of x in Γ is* 1*;*

(3) *for every vertex $x \in V(\mathfrak{G})$, the out-degree of x in Γ is at most* 1*;*

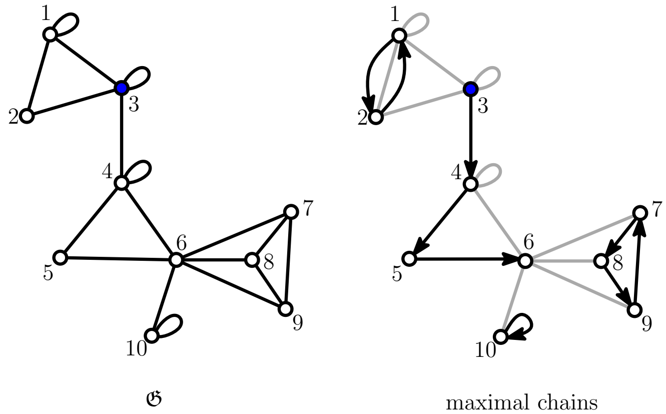

Figure 2.1 The loop graph $\mathfrak{G}$ for Example 2.2.3 and its maximal chains

(4) *each maximal chain is either a directed* 1*-cycle, a directed* 2*-cycle, a directed path, or a directed odd cycle given by rule (b), where an isolated vertex without any arcs on it is considered a directed* 1*-path;*

(5) B *is the set of tails of the directed paths in* Γ*.*

Proof. Since B is a ZFS-Z_{oc}, each vertex in B is blue initially and each vertex outside B turns blue exactly once, implying (1) and (2). A directed odd cycle given by rule (b) always forms a component itself in Γ, and each vertex of it has out-degree 1. If $x \to y_1$ and $x \to y_2$ are arcs in Γ, then y_1 and y_2 are two white neighbors of x and rule (a) cannot apply. Therefore (3) holds. For (4), since every vertex of Γ has in-degree at most 1 and out-degree 1, Γ is a disjoint union of directed cycles and directed paths. Since each directed n-cycle with $n \geq 3$ cannot be obtained by rule (a), it must be a directed odd cycle given by rule (b). Finally, for (5), B corresponds to those vertices with in-degree 0, which is the set of tails of each directed path in Γ. □

In contrast to the color-change rule on simple graphs, a vertex does not need to be blue to start its force. Let $(X_i \to Y_i)_{i=1}^{s}$ be a chronological list for Z_{oc} on a loop graph $\mathfrak{G}$. By Proposition 2.2.4, each vertex $x \in V(\mathfrak{G})$ is in at most one X_i for some i. If $x \in X_i$ is blue already before $X_i \to Y_i$ applies, or $x \notin X_i$ for any i, then x is said to be *blue-first.*

Proposition 2.2.5. *Let ζ be a ZFP-Z_{oc} on a loop graph $\mathfrak{G}$ and π a maximal chain. Then the following properties hold:*

(1) *if π is a directed 1-cycle on the vertex x, then x is not blue-first and has a loop in $\mathfrak{G}$;*

(2) *if π is a directed 2-cycle, then one of its two vertices is blue-first while the other is not, and the vertex which is not blue-first has no loop in $\mathfrak{G}$;*

(3) *if π is a directed odd cycle given by rule (b), then every vertex of π is not blue-first and has no loop in $\mathfrak{G}$;*

(4) *if x, y are in different maximal chains and x, y are not blue-first, then there is no edge between x and y in $\mathfrak{G}$.*

Proof. Directed 1-cycles and directed 2-cycles can only be given by rule (a). If π is a directed 1-cycle given by $\{x\} \to \{x\}$ in the chronological list, then x is not yet blue when this happens, and x has a loop in $\mathfrak{G}$ by rule (a). If π is a directed 2-cycle given by $\{x\} \to \{y\}$ first and $\{y\} \to \{x\}$ later in the chronological list, then x is not blue-first while y is, and x has no loop in $\mathfrak{G}$ by rule (a). Rule (b) gives (3) immediately. If x and y are as in (4) but x is adjacent to y in $\mathfrak{G}$, then neither of them can turn blue, a contradiction. □

Proposition 2.2.6. *If $(X_i \to Y_i)_{i=1}^{s}$ is a chronological list for Z_{oc} on $\mathfrak{G}$, then for $i < j$ there are no edges between X_i and Y_j.*

Proof. At the i-th step, Y_j is not yet blue, since $i < j$. Suppose there is an edge between X_i and Y_j. Then Y_j provides extra white neighbors to X_i other than Y_i. Thus X_i would not have a unique white neighbor, nor be an isolated loopless odd cycle. Hence $X_i \to Y_i$ is impossible, a contradiction. □

Proposition 2.2.7. *Let ζ be a ZFP-Z_{oc} on $\mathfrak{G}$ with initial blue set B and chronological list $(X_i \to Y_i)_{i=1}^s$. Then $B = V(\mathfrak{G}) \setminus \bigcup_{i=1}^s Y_i$. Also, $(Y_i \to X_i)_{i=s}^1$ is again a ZFP-Z_{oc}, starting with the initial blue set $B' = V(\mathfrak{G}) \setminus \bigcup_{i=1}^s X_i$. And B' is also a ZFS-Z_{oc} on $\mathfrak{G}$ with $|B| = |B'|$. This new zero forcing process is the reversal of ζ.*

Proof. The initial blue set B of a chronological list is those vertices not being changed to blue, so $B = V(\mathfrak{G}) \setminus \bigcup_{i=1}^s Y_i$. By Proposition 2.2.4, X_i's are mutually disjoint sets, and so are the Y_i's. Also, $|X_i| = |Y_i|$ for each i by definition. Hence $|B| = |B'|$ by the choice of B'. To see the reversal works, we claim that $Y_i \to X_i$ is a legal move under CCR-Z_{oc} when $B' = \bigcup_{j=i+1}^s X_j$ is all blue. At this situation, $\bigcup_{j=1}^i X_j$ is the set of white vertices, and Proposition 2.2.6 states that X_i is the only white set connected to Y_i. Therefore, $Y_i \to X_i$ works consecutively from s to 1. □

We note that the proof of Proposition 2.2.7 is analogous to that in [3] for simple graphs.

Theorem 2.2.8. *For any loop graph $\mathfrak{G}$ and any field F with $\operatorname{char} F \neq 2$, $M^F(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$.*

Proof. Since the color change rules for $Z(\mathfrak{G})$ are a subset of the color change rules for $Z_{oc}(\mathfrak{G})$, $Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$.

Let $n = |V(\mathfrak{G})|$, $k = Z_{oc}(\mathfrak{G})$, and B a ZFS-Z_{oc} of cardinality k. Also let $(X_i \to Y_i)_{i=1}^s$ be the corresponding chronological list with s steps. Let $A \in S^F(\mathfrak{G})$ be a matrix with $\operatorname{null}(A) = M^F(\mathfrak{G})$. Apply row/column permutations on A so that the columns follow the order of X_i's and the rows follow the order of Y_i's, and put all remaining columns to the right and rows to the bottom. Note that the permutations will not change the rank, but the new matrix will be of the form

$$\begin{bmatrix} A[Y_1,X_1] & & & ? & ? \\ O & A[Y_2,X_2] & & & ? \\ \vdots & & \ddots & & \vdots \\ O & \cdots & O & A[Y_s,X_s] & \\ ? & ? & \cdots & & ? \end{bmatrix},$$

where $A[Y_j, X_i]$ is the submatrix of A induced by rows in Y_j and columns in X_i.

This contains an upper-triangular block matrix, since Proposition 2.2.6 ensures that $A[Y_j, X_i] = O$ if $i < j$. Every diagonal block $A[Y_i, X_i]$ is either a 1×1 nonzero matrix, or a matrix described by a loopless odd cycle, which is nonsingular by Remark 2.1.3. This means the rank of A is at least $|\bigcup_i^s Y_i| = n - k$. Therefore $M^F(\mathfrak{G}) = \text{null}(A) \leq k = Z_{oc}(\mathfrak{G})$. □

The proof of Theorem 2.2.8 is based on Remark 2.1.3, and that is why we need char $F \neq 2$.

Corollary 2.2.9. *For any loop configuration $\mathfrak{G}$ of a complete graph or a cycle, $M^{\mathbb{R}}(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$.*

Proof. If there are at least 3 nonloop vertices $\{x, y, z\}$ on a loop configuration $\mathfrak{G}$ of a complete graph, then $V(\mathfrak{G}) \setminus \{x, y, z\}$ is a ZFS-Z_{oc} on $\mathfrak{G}$. If $\mathfrak{G}$ is a loopless odd cycle, then the empty set is a ZFS-Z_{oc} on $\mathfrak{G}$. Together with Proposition 2.1.1 and Proposition 2.1.2, $M^{\mathbb{R}}(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ for these loop graphs. □

We end this section with Example 2.2.10, showing the gap $Z(\mathfrak{G}) - Z_{oc}(\mathfrak{G})$ can be arbitrarily large for loop graphs.

Example 2.2.10. Let $G_n = K_1 \vee (nK_3)$ be the simple graph defined as the join of a vertex and n copies of K_3, the complete graph on 3 vertices. Figure 2.2 shows G_2. Let $\mathfrak{G}_n^0$ be the loop configuration of G_n without any loop and $x \in V(\mathfrak{G}_n^0)$ the vertex

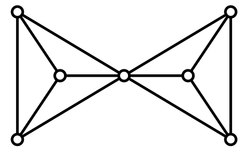

Figure 2.2 The graph $K_1 \bigvee (2K_3)$ for Example 2.2.10

adjacent to all other vertices. Then $Z_{oc}(\mathfrak{G}_n^0) = 1$, since $\{x\}$ is a ZFS-Z_{oc} on $\mathfrak{G}_n^0$. However, $Z(\mathfrak{G}_n^0) = n+1$. To see this, observe that $\mathfrak{G}_n^0 - x$ is n copies of the loopless odd cycle $\mathfrak{C}_3^0$. In $\mathfrak{G}_n^0$ each copy of $\mathfrak{C}_3^0$ needs at least one blue vertex, for otherwise there is no way to turn this copy blue; taking only these n blue vertices does not allow forcing to begin, but these n vertices along with x becomes a (conventional) zero forcing set on $\mathfrak{G}_n^0$. So $n+1$ blue vertices is the minimum requirement. Also, $M^{\mathbb{R}}(\mathfrak{G}_n^0) = 1 = Z_{oc}(\mathfrak{G}_n^0)$, since it does not contain a unique spanning composite cycle (see [7]).

2.3 Enhanced odd cycle zero forcing number $\widehat{Z}_{oc}(G)$

The enhanced zero forcing number demonstrates that an upper bound for loop graphs can lead to an upper bound for simple graphs. This also applies to the odd cycle zero forcing number.

Definition 2.3.1. Let G be a simple graph. Running over all loop configurations $\mathfrak{G}$ of G, the *enhanced odd cycle zero forcing number* $\widehat{Z}_{oc}(G)$ for the simple graph G is

$$\widehat{Z}_{oc}(G) = \max_{\mathfrak{G}} Z_{oc}(\mathfrak{G}).$$

The proof of the next theorem follows that of Corollary 2.24 in [4].

Theorem 2.3.2. *For any simple graph G and any field F with $\operatorname{char} F \neq 2$, $M^F(G) \leq \widehat{Z}_{oc}(G) \leq \widehat{Z}(G)$.*

Proof. Let A be a matrix in $\mathcal{S}^F(G)$ such that $\operatorname{null}(A) = M^F(G)$. Following the zero-nonzero pattern on the diagonal entries of A, A must fall into $\mathcal{S}^F(\mathfrak{G})$ for some loop configuration $\mathfrak{G}$ of G. As a consequence,

$$M^F(G) = \text{null}(A) \le M^F(\mathfrak{G}) \le Z_{oc}(\mathfrak{G}) \le \widehat{Z}_{oc}(G),$$

by Theorem 2.2.8. And again by Theorem 2.2.8, $Z_{oc}(\mathfrak{G}) \le Z(\mathfrak{G})$, so $\widehat{Z}_{oc}(G) \le \widehat{Z}(G)$ by definitions. □

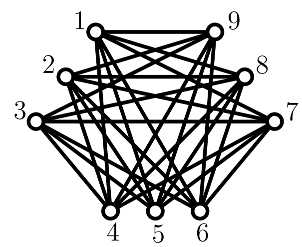

Figure 2.3 Labeled $K_{3,3,3}$

Example 2.3.3. Let G be the complete tripartite graph $K_{3,3,3}$ (as a simple graph), which is shown in Figure 2.3. For this graph, we show that $Z(G) = 7 = \widehat{Z}(G)$ but $\widehat{Z}_{oc}(G) = 6 = M^{\mathbb{R}}(G)$.

We start by showing $Z(G) = 7 = \widehat{Z}(G)$. First consider the simple graph G and a zero forcing set B on G. If $|B \cap \{1,2,3\}| < 2$, then there is no way to turn all of $\{1,2,3\}$ blue. So each of the clusters $\{1,2,3\}$, $\{4,5,6\}$, and $\{7,8,9\}$ contains at least 2 blue vertices. But 6 vertices with 2 in each clusters is not enough to make $V(G)$ all blue. Therefore, $\{1,2,3,4,5,7,8\}$ is a minimum zero forcing set on G, and $Z(G) = 7$. This same argument also works for the zero forcing number of the loop graph $\mathfrak{G}^0$, where $\mathfrak{G}^0$ is the loop configuration of G without any loop. So $7 = Z(\mathfrak{G}^0) \le \widehat{Z}(G) \le Z(G) = 7$ and $\widehat{Z}(G) = 7$ also.

Next we show $\widehat{Z}_{oc}(G) = 6 = M^{\mathbb{R}}(G)$. Let $\mathfrak{G}$ be a loop configuration of $K_{3,3,3}$. Assume each vertex in $\{1,2,3\}$ has a loop. Then the initial blue set $B = \{4,5,6,7,8,9\}$ can make all vertices blue by rule (a), so $Z_{oc}(\mathfrak{G}) \le 6$ in this case. Similarly, $\{1,2,3\}$ can be replaced by the other clusters $\{4,5,6\}$ and $\{7,8,9\}$. So now assume at least one vertex in each cluster does not have a loop, say 1, 4, and 7. In this case, $\{2,3,5,6,8,9\}$ forms a ZFS-Z_{oc}, since $\{1,4,7\}$ forms a loopless odd cycle and rule (b) applies. Throughout all cases, $\widehat{Z}_{oc}(G) \le 6$. On the other hand, $M^{\mathbb{R}}(G) \ge 6$, because its adjacency matrix over $\mathbb{R}$ has nullity 6. Therefore, $\widehat{Z}_{oc}(G) = 6 = M^{\mathbb{R}}(G)$.

Finally, if we consider the adjacency matrix of G over $\mathbb{F}_2$, the field of two elements, then its nullity is 7 instead of 6. So $7 \le M^{\mathbb{F}_2}(G) \le \widehat{Z}(G) = 7$. The discrepancy between $\widehat{Z}(G)$ and $\widehat{Z}_{oc}(G)$ is because $\widehat{Z}(G) \ge M^F(G)$ works for an arbitrary field F, but $\widehat{Z}_{oc}(G) \ge M^F(G)$ works only when $\operatorname{char} F \ne 2$.

2.4 Graph blowups

The simple graph $K_{3,3,3}$ in Example 2.3.3 demonstrates a relation between the zero forcing number for simple graphs and that for loop graphs. Let $\mathfrak{C}_3^0$ be the loopless odd cycle on 3 vertices. The simple graph $K_{3,3,3}$ can be viewed as the simple graph obtained from $\mathfrak{C}_3^0$ by replacing each vertex with a cluster of $t = 3$ isolated vertices and replacing each edge with a complete bipartite graph joining the corresponding clusters. Example 2.3.3 satisfies

$$Z(K_{3,3,3}) = \widehat{Z}(K_{3,3,3}) = (t-1) \times |V(\mathfrak{C}_3^0)| + Z(\mathfrak{C}_3^0)$$

and

$$\widehat{Z}_{oc}(K_{3,3,3}) = (t-1) \times |V(\mathfrak{C}_3^0)| + Z_{oc}(\mathfrak{C}_3^0)$$

with $t = 3$.

The transformation of $\mathfrak{C}_3^0$ to $K_{3,3,3}$ is called a *blowup*. In this section, we discuss how graph blowups can bridge loop graphs and simple graphs.

Definition 2.4.1. Let $\mathfrak{G}$ be a loop graph with $V(\mathfrak{G}) = \{v_i\}_{i=1}^n$, and $(t_1, t_2, \dots, t_n)$ a sequence of n positive integers. The $(t_1, t_2, \dots, t_n)$-*blowup* of $\mathfrak{G}$ is the simple graph

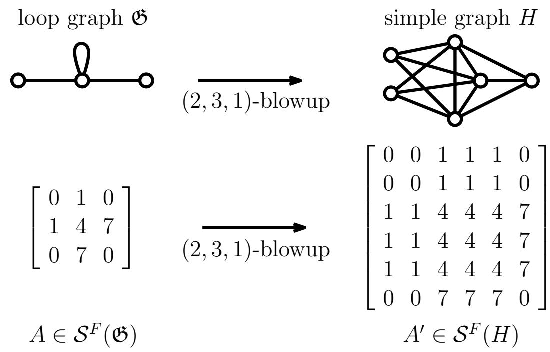

Figure 2.4 An illustration of the blowup of graphs and matrices

obtained from $\mathfrak{G}$ by replacing v_i with a cluster V_i of t_i vertices and for each edge v_iv_j ($i = j$ is possible), joining every vertex of V_i with every vertex of V_j.

Definition 2.4.2. Let $A = [a_{i,j}]$ be a symmetric matrix indexed by $\{v_i\}_{i=1}^n$. and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers. Denote $n' = \sum_{i=1}^n t_i$. The $(t_1, t_2, \ldots, t_n)$-*blowup* of A is the $n' \times n'$ matrix obtained from A by replacing the i,j-entry $a_{i,j}$ of A with $a_{i,j}J_{t_i,t_j}$, where J_{t_i,t_j} is the $t_i \times t_j$ all one matrix.

These definitions are illustrated in Figure 2.4.

Lemma 2.4.3. *Let $\mathfrak{G}$ be a loop graph with $V(\mathfrak{G}) = \{v_i\}_{i=1}^n$, and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. Then $\mathrm{mr}^F(H) \le \mathrm{mr}^F(\mathfrak{G})$ and $M^F(H) \ge \sum_{i=1}^n (t_i - 1) + M^F(\mathfrak{G})$ for any field F.*

Proof. Let A be a matrix in $\mathcal{S}^F(\mathfrak{G})$ and A' the $(t_1, t_2, \ldots, t_n)$-blowup of A. Then A' is a matrix in $\mathcal{S}^F(H)$. Also, since deleting repeated rows and columns does not change the rank, $\mathrm{rank}(A) = \mathrm{rank}(A')$. Therefore, $\mathrm{mr}^F(H) \le \mathrm{mr}^F(\mathfrak{G})$ and $M^F(H) \ge \sum_{i=1}^n (t_i - 1) + M^F(\mathfrak{G})$ for any field F. □

Lemma 2.4.4. *Let $\mathfrak{G}$ be a loop graph on n vertices and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers with $t_i \ge 2$ for all i. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. Then*

$$Z(H) = Z(\mathfrak{H}') = \sum_{i=1}^n (t_i - 1) + Z(\mathfrak{G}),$$

where $\mathfrak{H}'$ is the loop configuration of H such that every vertex in a cluster corresponding to a clique has a loop while the others do not have a loop.

The second equality also holds if Z is replaced by Z_{oc}. That is,

$$Z_{oc}(\mathfrak{H}') = \sum_{i=1}^{n}(t_i - 1) + Z_{oc}(\mathfrak{G}).$$

Proof. We consider $Z(H)$ first.

Let $V(\mathfrak{G}) = \{v_i\}_{i=1}^n$ and V_i the cluster of t_i vertices. On the simple graph H, if there are two white vertices in a cluster V_i, then there is no way to make V_i all blue. So in order to be a zero forcing set on H, each V_i has at most one white vertex. This ensures $Z(H) \geq \sum_{i=1}^n (t_i - 1)$. Say the cluster V_i is *blue* if all its vertices is blue, and V_i is *one-white* if one of its vertices is white. Then $Z(H)$ will be $\sum_{i=1}^n (t_i - 1)$ plus the minimum number of blue clusters.

Assume each V_i is either blue or one-white. Denote $V_i \to V_j$ if $x \to y$ happens on H for some $x \in V_i$ and $y \in V_j$. Since $t_i \geq 2$, each one-white cluster contains at least one blue vertex. Suppose v_i has no loop in $\mathfrak{G}$, then $V_i \to V_j$ on H does not require V_i to be blue; similarly, $v_i \to v_j$ on $\mathfrak{G}$ does not require v_i to be blue. Suppose v_i has a loop in $\mathfrak{G}$, then $V_i \to V_i$ can happen when all other neighbors are blue already; this is the same case for $v_i \to v_i$. Therefore, when each cluster is either blue or one-white, $V_i \to V_j$ on H if and only if $v_i \to v_j$ on $\mathfrak{G}$. So the minimum number of blue clusters is $Z(\mathfrak{G})$ and

$$Z(H) = \sum_{i=1}^{n}(t_i - 1) + Z(\mathfrak{G}).$$

The same argument works when H is replaced by $\mathfrak{H}'$. It also works when rule (b) comes in. Suppose that $\{v_i\}_{i \in \alpha}$ forms a loopless odd cycle on $\mathfrak{G}$ for some index set α and rule (b) can be applied on it. Then at this step each cluster in $\{V_i\}_{i \in \alpha}$ is one-white and the only white vertices in each of them form a loopless odd cycle on $\mathfrak{H}'$. So $Z_{oc}(\mathfrak{H}') = \sum_{i=1}^n (t_i - 1) + Z_{oc}(\mathfrak{G})$ holds. □

Lemma 2.4.5. *Let $\mathfrak{G}$ be a loop graph on n vertices and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers with $t_i \geq 3$ for all i. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. Then*

$$\widehat{Z}_{oc}(H) = \sum_{i=1}^{n}(t_i - 1) + Z_{oc}(\mathfrak{G}).$$

Proof. Let $h = \sum_{i=1}^{n}(t_i - 1) + Z_{oc}(\mathfrak{G})$. By Lemma 2.4.4, at least one loop configuration $\mathfrak{H}'$ of H has $Z_{oc}(\mathfrak{H}') = h$, so it is enough to show that any loop configuration $\mathfrak{H}$ of H has $Z_{oc}(\mathfrak{H}) \leq h$.

Let $\mathfrak{H}$ be a loop configuration of H and B be a minimum ZFS-Z_{oc} of $\mathfrak{G}$. We adopt the notation from Lemma 2.4.4. We mark V_i blue if $v_i \in B$ and one-white if $v_i \notin B$; whenever a cluster V_i is marked one-white, we pick the only white vertex to be a nonloop vertex, unless every vertex in V_i has a loop. Call this set B'. Starting with B', we can do the corresponding forces $V_i \to V_j$ whenever $v_i \to v_j$ happens in $\mathfrak{G}$. If rule (b) never applies in $\mathfrak{G}$, then B' is a ZFS-Z_{oc} of $\mathfrak{H}$ with $|B'| = h$ and we are done. So assume rule (b) first happens at some step, and it applies to a loopless odd cycle $\mathfrak{C}$ on $\mathfrak{G}$. Denote $V(\mathfrak{C}) = \{v_i\}_{i\in\alpha}$ for some index set α. If every cluster V_i in $\{V_i\}_{i\in\alpha}$ contains a nonloop vertex on the loop configuration $\mathfrak{H}$, then by the choice of B' there is a loopless odd cycle on $\mathfrak{H}$ and the process continues. Now assume at least one cluster V_a has all its vertices with loops. Say v_b and v_c are the two neighbors of v_a in $\mathfrak{C}$. We modify B' by marking V_b and V_c blue, and setting all vertices in V_a as white. This modification does not increase the number of blue vertices, since marking V_b and V_c blue add two blue vertices, but setting all V_a white loses at least two blue vertices by the fact $t_a \geq 3$. Note that V_a is an independent set since $\mathfrak{C}$ is loopless, and all its vertices has loops. By starting with the new B', the same process can go on until rule (b) applies to $\mathfrak{C}$. At this step, v_a has only two white neighbors v_b and v_c; this means at the stage where rule (b) was applied in $\mathfrak{G}$ every vertex in V_a can force itself blue, since V_b and V_c are blue initially. Now by applying rule (a) only, every cluster in $\{V_i\}_{i\in\alpha}$ turns blue eventually, and the process continues. Since all loopless odd cycles given by rule (b) are mutually isolated by Proposition 2.2.5, we can do the modification separately, and find a ZFS-Z_{oc} of $\mathfrak{H}$ with cardinality less than or equal to h. Therefore, $Z_{oc}(\mathfrak{H}) \le h$ and $\widehat{Z}_{oc}(H) = h$. □

Remark 2.4.6. In Lemma 2.4.5, the assumption $t_i \ge 3$ for all i can be relaxed to $t_i \ge 3$ whenever v_i has no loop in $\mathfrak{G}$ and $t_i \ge 2$ otherwise.

Theorem 2.4.7. *Let $\mathfrak{G}$ be a loop graph on n vertices and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. If $M^F(\mathfrak{G}) = Z(\mathfrak{G})$ for some field F and $t_i \ge 2$ for all i, then*

$$M^F(H) = Z(H) = \widehat{Z}(H) = \sum_{i=1}^{n}(t_i - 1) + M^F(\mathfrak{G}).$$

If $M^F(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ for some field F with $\operatorname{char} F \ne 2$ *and $t_i \ge 3$ for all i, then*

$$M^F(H) = \widehat{Z}_{oc}(H) = \sum_{i=1}^{n}(t_i - 1) + M^F(\mathfrak{G}).$$

Proof. This immediately comes from Lemma 2.4.3, Lemma 2.4.4, Lemma 2.4.5, and Theorem 2.2.8. □

Corollary 2.4.8. *Let G be a tree, a cycle, or a complete graph, and $\mathfrak{G}$ its loop configuration with $V(\mathfrak{G}) = \{v_i\}_{i=1}^{n}$. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$ with $t_i \ge 3$ for all i. Then $M^{\mathbb{R}}(H) = \widehat{Z}_{oc}(H)$. Moreover, $M^{\mathbb{R}}(H) = Z(H)$ if G is a tree.*

Proof. If G is a tree, then $M^{\mathbb{R}}(\mathfrak{G}) = Z(\mathfrak{G})$ [5]; if G is a cycle or a complete graph, then $M^{\mathbb{R}}(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ by Corollary 2.2.9. By applying Theorem 2.4.7, the equality holds. □

Example 2.3.3 together with Lemma 2.4.4 and Theorem 2.4.7 also provide a family of simple graphs with large $\widehat{Z}(G) - \widehat{Z}_{oc}(G)$.

Corollary 2.4.9. *Let $G_n = K_1 \vee (nK_3)$ be the simple graph in Example 2.2.10 and $\mathfrak{G}_n^0$ the loop configuration of G_n without any loop. Let H_n be the $(3, 3, \ldots, 3)$-blowup of G_n. Then*

$$\widehat{Z}_{oc}(H_n) = 2 \cdot |V(\mathfrak{G}_n^0)| + 1 \text{ and } \widehat{Z}(H_n) = 2 \cdot |V(\mathfrak{G}_n^0)| + n + 1.$$

2.5 Graph complement conjecture for $\widehat{Z}_{oc}(G)$

The *graph complement conjecture* for the maximum nullity (GCC-M) for simple graphs [2] states that

$$M^F(G) + M^F(\overline{G}) \geq n - 2,$$

where $\overline{G}$ is the complement of G. Corollary 2.5.1 below shows GCC-M is true for most graph blowups.

Corollary 2.5.1. *Let $\mathfrak{G}$ be a loop graph and H the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. If $t_i \geq 2$ for all i, then GCC-M is true for H over any field F. That is,*

$$M^F(H) + M^F(\overline{H}) \geq |V(H)| - 2.$$

Proof. Notice that $\overline{H}$ is the $(t_1, t_2, \ldots, t_n)$-blowup of $\overline{\mathfrak{G}}$, where $\overline{\mathfrak{G}}$ is the complement of $\mathfrak{G}$ as loop graphs; that is, there is an edge (or a loop) between v_i and v_j in $\overline{\mathfrak{G}}$ if and only if there is no edge (or no loop) between v_i and v_j in $\mathfrak{G}$.

By Lemma 2.4.3, $M^F(H) \geq \frac{1}{2}|V(H)|$ since $t_i \geq 2$ for all i. Similarly, $M^F(\overline{H}) \geq \frac{1}{2}|V(H)|$. So

$$M^F(H) + M^F(\overline{H}) \geq |V(H)| > |V(H)| - 2,$$

and GCC-M holds for H. □

If β is a graph parameter for simple graphs, the graph complement conjecture for β (GCC-β) is stated as

$$\beta(G) + \beta(\overline{G}) \geq n - 2.$$

In [9], GCC-tw, GCC-Z_+, and GCC-Z are proven to be true, where tw is the *tree-width*, Z_+ is the *positive semidefinite zero forcing number*, and Z is the zero forcing number for simple graphs. The relation between these parameters can be found in Fig. 1.1 of [4].

We claim that GCC-Z_+ implies GCC-$\widehat{Z}_{oc}$, so GCC-$\widehat{Z}_{oc}$ is also true. We need an intermediate parameter. The *loop zero forcing number* $Z_\ell(G)$ for simple graphs G is defined as $Z(\mathfrak{G})$, where $\mathfrak{G}$ is the loop configuration of G such that isolated vertices have no loop while the others have a loop [4]. Since rule (b) can never apply on $\mathfrak{G}$, $Z(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ and $Z_\ell(G) = Z_{oc}(\mathfrak{G}) \leq \widehat{Z}_{oc}(G)$. Also, it is known that $Z_+(G) \leq Z_\ell(G)$ [4]. Therefore, $Z_+(G) \leq \widehat{Z}_{oc}(G)$ for every simple graph G, which means GCC-Z_+ implies GCC-$\widehat{Z}_{oc}$.

2.6 Examples with $\widehat{Z}_{oc}(G) - M^{\mathbb{R}}(G)$ large

A 5-*sun*, H_5, is a simple graph obtained from C_5 by appending a leaf to each vertex. It is known that $M^{\mathbb{R}}(G) = 2 = \widehat{Z}(G)$ but $Z(G) = 3$ [1, 4]. Thus $\widehat{Z}_{oc}(G) = 2$, by Theorem 2.2.8. To get a discrepancy between $\widehat{Z}_{oc}(G)$ and $M^{\mathbb{R}}(G)$, we insert one more leaf to each of the leaves of H_5 and call it a *long* 5-*sun*, denoted as LH_5. A *long* 5-*sun sequence* of length n is the simple graph shown in Figure 2.5, which concatenates n copies of LH_5. Proposition 2.6.1 shows that for this family of graphs and hence in general for simple graphs, $\widehat{Z}_{oc}(G) - M^{\mathbb{R}}(G)$ can be arbitrarily large.

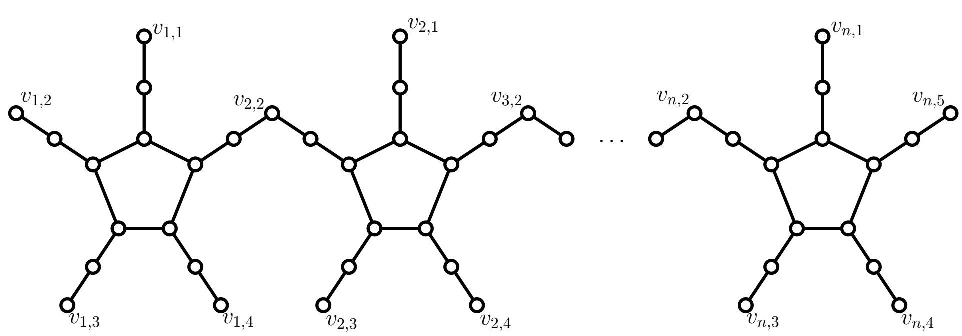

Figure 2.5 A sequence of long 5-sun

Proposition 2.6.1. *Let L_n be the long 5-sun sequence of length n described above. Then $M^{\mathbb{R}}(L_n) = n + 1$ and $\widehat{Z}_{oc}(L_n) = \widehat{Z}(L_n) = Z(L_n) = 2n + 1$.*

Proof. The cut-vertex reduction formula [6] states that

$$M^{\mathbb{R}}(G_1 \oplus_v G_2) = \max\{M^{\mathbb{R}}(G_1) + M^{\mathbb{R}}(G_2) - 1, M^{\mathbb{R}}(G_1 - v) + M^{\mathbb{R}}(G_2 - v) - 1\},$$

where $G_1 \oplus_v G_2$ is obtained from G_1 and G_2 by identifying the vertex v on each of them. Suppose x is a leaf on a graph G and y is the only neighbor of x. Then by applying the formula on y, one immediate observation is $M^{\mathbb{R}}(G) \geq M^{\mathbb{R}}(G-x)$; additionally, if y is of degree 2, then $M^{\mathbb{R}}(G) = M^{\mathbb{R}}(G-x)$. Therefore, $M^{\mathbb{R}}(L_1) = M^{\mathbb{R}}(H_5) = 2$. Now write L_n as $L_{n-1} \oplus_v L_1$, where v is the vertex $v_{n-1,5}$ in L_{n-1} and a leaf in L_1. Since v is a leaf on L_{n-1} and on L_1, the observation reduces the formula to be $M^{\mathbb{R}}(L_n) = M^{\mathbb{R}}(L_{n-1}) + M^{\mathbb{R}}(L_1) - 1 = M^{\mathbb{R}}(L_{n-1}) + 1$. Inductively, $M^{\mathbb{R}}(L_n) = n + 1$.

For zero forcing numbers, the set $\{v_{1,1}, v_{1,2}, v_{1,3}\} \bigcup \{v_{i,1}, v_{i,3}\}_{i=2}^{n}$ labeled in Figure 2.5 forms a zero forcing set on the simple graph L_n. So $Z(L_n) \leq 2n + 1$.

On the other hand, we show $Z_{oc}(\mathfrak{L}_n^{\ell}) = 2n + 1$, where $\mathfrak{L}_n^{\ell}$ is the loop configuration of L_n so that each vertex has a loop. First we make some observations. By Proposition 2.2.5, the maximal chains on $\mathfrak{L}_n^{\ell}$ can only be directed 1-cycles or directed paths, and the number of directed paths is the cardinality of the ZFS-Z_{oc}. Even more, there are no edges between any two distinct directed 1-cycles. Let $x \in V(\mathfrak{L}_n^{\ell})$ be a pendent vertex, which means x has only one neighbor y other than itself. Let π_x and π_y be the maximal chain containing x and y respectively, where $\pi_x = \pi_y$ is possible. By the structure of $\mathfrak{L}_n^{\ell}$, if π_x is a directed path, then x is an endpoint of π_x; if π_x is a directed 1-cycle, then π_y must be a directed path and y is an endpoint of π_y. In either case, $\{x, y\}$ must contain an endpoint for some maximal chain. Now we claim $Z_{oc}(\mathfrak{L}_n^{\ell}) \geq 2n+1$ by induction on n. For $n = 1$, there are 5 pendent vertices in $\mathfrak{L}_1^{\ell}$. Each directed path has only 2 endpoints, so $\lceil \frac{5}{2} \rceil = 3$ directed paths are needed. Assume $Z_{oc}(\mathfrak{L}_{n-1}^{\ell}) \geq 2n - 1$. Note that $\mathfrak{L}_n^{\ell}$ is obtained from $\mathfrak{L}_{n-1}^{\ell}$ by attaching the last copy of $\mathfrak{L}_1^{\ell}$, where $V(\mathfrak{L}_{n-1}^{\ell}) \bigcap V(\mathfrak{L}_1^{\ell}) = \{v_{n,2}\}$. There are still 4 pendent vertices on $V(\mathfrak{L}_1^{\ell})$. Only one of the 4 vertices can combine with a directed path from $\mathfrak{L}_{n-1}^{\ell}$. So at least $2n - 1 + \lceil \frac{4-1}{2} \rceil = 2n + 1$ directed paths are needed for $\mathfrak{L}_n^{\ell}$. This means

$$2n + 1 \leq Z_{oc}(\mathfrak{L}_n^{\ell}) \leq \widehat{Z}_{oc}(L_n) \leq \widehat{Z}(L_n) \leq Z(L_n) \leq 2n + 1.$$

Hence every inequality is an equality. □

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CHAPTER 3. USING A NEW ZERO FORCING PROCESS TO GUARANTEE THE STRONG ARNOLD PROPERTY

A paper published in Linear Algebra and its Applications

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Abstract

The maximum nullity $M(G)$ and the Colin de Verdière type parameter $\xi(G)$ both consider the largest possible nullity over matrices in $\mathcal{S}(G)$, which is the family of real symmetric matrices whose i,j-entry, $i \neq j$, is nonzero if i is adjacent to j, and zero otherwise; however, $\xi(G)$ restricts to those matrices A in $\mathcal{S}(G)$ with the Strong Arnold Property, which means $X = O$ is the only symmetric matrix that satisfies $A \circ X = O$, $I \circ X = O$, and $AX = O$. This paper introduces zero forcing parameters $Z_{\text{SAP}}(G)$ and $Z_{\text{vc}}(G)$, and proves that $Z_{\text{SAP}}(G) = 0$ implies every matrix $A \in \mathcal{S}(G)$ has the Strong Arnold Property and that the inequality $M(G) - Z_{\text{vc}}(G) \leq \xi(G)$ holds for every graph G. Finally, the values of $\xi(G)$ are computed for all graphs up to 7 vertices, establishing $\xi(G) = \lfloor Z \rfloor(G)$ for these graphs.

3.1 Introduction

A *minimum rank problem* for a graph G is to determine what is the smallest possible rank, or equivalently the largest possible nullity, among a family of matrices associated with G. One classical way to associate matrices to a graph G is through $\mathcal{S}(G)$, which is defined as the set of all real symmetric matrices whose i,j-entry, $i \neq j$, is nonzero whenever i and j are adjacent in G, and zero otherwise. Note that the diagonal entries can be any real number. Another association is $\mathcal{S}_+(G)$, which is the set of positive semidefinite matrices in $\mathcal{S}(G)$. Thus, the *maximum nullity* $M(G)$ and the *positive semidefinite maximum nullity* $M_+(G)$ are defined as

$$M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\}, \text{ and}$$
$$M_+(G) = \max\{\text{null}(A) : A \in \mathcal{S}_+(G)\}.$$

The classical minimum rank problem is a branch of the *inverse eigenvalue problem*, which asks for a given multi-set of real numbers, is there a matrix in $\mathcal{S}(G)$ such that its spectrum is composed of these real numbers. If λ is an eigenvalue of some matrix $A \in \mathcal{S}(G)$, then its multiplicity should be no higher than $M(G)$, for otherwise $A - \lambda I$ has nullity higher than $M(G)$. Similarly, $M_+(G)$ provides an upper bound for the multiplicities of the smallest and the largest eigenvalues. Also, $M_+(G)$ is closely related to faithful *orthogonal representations* [12].

Other families of matrices are defined through the Strong Arnold Property. A matrix A is said to have the *Strong Arnold Property* (or SAP) if the zero matrix is the only symmetric matrix X that satisfies the three conditions $A \circ X = O$, $I \circ X = O$, and $AX = O$. Here I and O are the identity matrix and the zero matrix of the same size as A, respectively, and $\circ$ is the Hadamard (entrywise) product of matrices. By adding the SAP to the conditions of the abovementioned families, the *Colin de Verdière type parameters* are defined as

$$\xi(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ has the SAP}\} \text{ [5], and}$$
$$\nu(G) = \max\{\text{null}(A) : A \in \mathcal{S}_+(G), A \text{ has the SAP}\} \text{ [8].}$$

These parameters are variations of the original Colin de Verdière parameter $\mu(G)$ [7], which is defined as the maximum nullity over matrices A such that

- $A \in \mathcal{S}(G)$ and every off-diagonal entry of A is non-positive (called a *generalized Laplacian of* G),
- A has exactly one negative eigenvalue including the multiplicity, and
- A has the SAP.

In order to see how the SAP makes a difference between these parameters, we define $M_\mu(G)$ as the maximum nullity of the same family of matrices by ignoring the SAP, i.e. the maximum nullity of a generalized Laplacian A of G such that A has exactly one negative eigenvalue.

The SAP gives $\xi(G)$, $\nu(G)$, and $\mu(G)$ nice properties. For example, they are *minor monotone* [12]. A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of deleting edges, deleting vertices, and contracting edges; a graph parameter ζ is said to be minor monotone if $\zeta(H) \leq \zeta(G)$ whenever H is a minor of G. By the graph minor theorem (e.g., see [10]), for a given integer d and a minor monotone parameter ζ, the minimal forbidden minors for $\zeta(G) \leq d$ consist of only finitely many graphs. Here ζ can be ξ, ν or μ. More specifically, $\mu(G) \leq 3$ if and only if G is a planar graph [18], which is characterized by the forbidden minors K_5 and $K_{3,3}$.

However, the SAP also makes the Colin de Verdière type parameters less controllable by the existing tools. For example, zero forcing parameters, which will be defined in Section 3.1.2, were used extensively as a bound for the minimum rank problem. For the classical zero forcing number $Z(G)$, it is known that $M(G) \leq Z(G)$ for all graphs [2]; and $M(G) = Z(G)$ when G is a tree or $|V(G)| \leq 7$ [2, 9]. An analogy for $\xi(G)$ is the minor monotone floor of the zero forcing number, which is denoted as $\lfloor Z \rfloor(G)$ and will be defined in Section 3.4. It is known that $\xi(G) \leq \lfloor Z \rfloor(G)$ for all graphs [4]. The similar statement $\xi(G) = \lfloor Z \rfloor(G)$ is not always true when G is a tree [4], and no results about $\xi(G)$ and $\lfloor Z \rfloor(G)$ for small graphs are known.

The main goal of this paper is to establish a connection between zero forcing parameters and the SAP, and derive consequences. This leads to some questions. Does some graph structure guarantee that every $A \in \mathcal{S}(G)$ has the SAP? Thus, the maximum nullity does not change when the SAP condition is added. Specifically, when A is a generalized Laplacian of some graph G and has exactly one negative eigenvalue, some graph structures do guarantee that A has the SAP [15,21]; however, general results on this problem remain unknown. On the other hand, is there a strategy to perturb any given matrix such that it guarantees the SAP? Thus, the rank changed by the perturbation gives an upper bound for $M(G) - \xi(G)$.

In Section 3.2, we introduce a new parameter $Z_{\text{SAP}}(G)$ and its variants Z^{ℓ}_{SAP} and Z^{+}_{SAP}, and prove in Theorem 3.2.6 that under the condition $Z_{\text{SAP}}(G) = 0$, every matrix $A \in \mathcal{S}(G)$ has the SAP. Thus, $\xi(G) = M(G)$, $\nu(G) = M_{+}(G)$, and $\mu(G) = M_{\mu}(G)$ when $Z_{\text{SAP}}(G) = 0$, so finding the values of Colin de Verdière type parameters is equivalent to finding the values of the corresponding parameters. Table 3.1 in Section 3.2.2 indicates that there are actually a considerable proportion of graphs that have this property.

In Section 3.3, another parameter $Z_{\text{vc}}(G)$ and its variant $Z^{\ell}_{\text{vc}}(G)$ are defined, and Theorem 3.3.2 states that $M(G) - \xi(G) \leq Z_{\text{vc}}(G)$ for every graph G. With the help of $Z_{\text{SAP}}(G)$, $Z_{\text{vc}}(G)$, and some existing theorems, Section 3.4 provides the result that $\xi(G) = \lfloor Z \rfloor(G)$ for graphs G up to 7 vertices.

All parameters introduced in this paper and their relations are illustrated in Figure 3.1. A brief description of the related theorems is given on the sides. A line between two parameters means the lower one is less than or equal to the upper one.

Throughout the paper, the neighborhood of a vertex i in a graph G is denoted as $N_G(i)$, while the closed neighborhood is denoted as $N_G[i]$, which equals $N_G(i) \cup \{i\}$. The induced subgraph on a vertex set W of G is denoted as $G[W]$. If A is a matrix, U

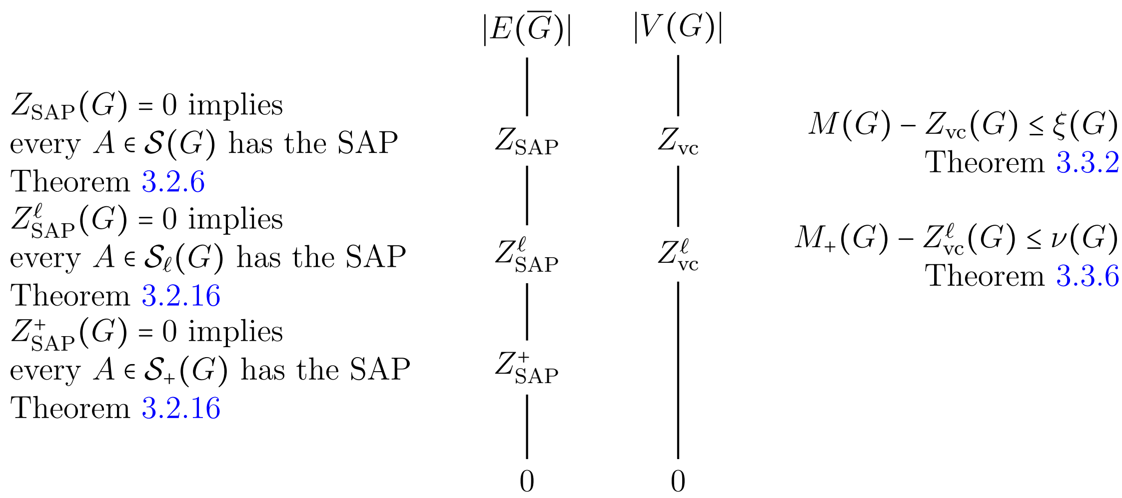


Figure 3.1 Parameters introduced in this paper

and W are subsets of the row and column indices of A respectively, then $A[U, W]$ is the submatrix of A induced on the rows of U and columns of W; if U and W are ordered sets, then permute the rows and columns of this submatrix accordingly.

3.1.1 SAP system and its matrix representation

Let G be a graph on n vertices, and $\overline{m} = |E(\overline{G})|$. In order to see if a matrix $A \in \mathcal{S}(G)$ has the SAP or not, the matrix X can be viewed as a symmetric matrix with $\overline{m}$ variables at the positions of non-edges so that X satisfies $A \circ X = I \circ X = O$. Next, $AX = O$ leads to n^2 restrictions on the $\overline{m}$ variables, which forms a linear system. Call this linear system the *SAP system of* A, which can also be written as an $n^2 \times \overline{m}$ matrix.

Definition 3.1.1. Let G be a graph on n vertices, $\overline{m} = |E(\overline{G})|$, and $A = \begin{bmatrix} a_{i,j} \end{bmatrix} \in \mathcal{S}(G)$. Given an order of the set of non-edges, the *SAP matrix of* A with respect to this order is an $n^2 \times \overline{m}$ matrix Ψ whose rows are indexed by pairs (i, k) and columns are indexed by the non-edges $\{j,h\}$ such that

$$\Psi_{(i,k),\{j,h\}} = \begin{cases} 0 & \text{if } k \notin \{j,h\}, \\ a_{i,j} & \text{if } k \in \{j,h\} \text{ and } k = h. \end{cases}$$

The rows follow the order $(i,k) < (j,h)$ if and only if $k < h$, or $k = h$ and $i < j$; the columns follow the order of the non-edges.

Remark 3.1.2. Let G be a graph, $A \in \mathcal{S}(G)$, and Ψ the SAP matrix of A with respect a given order of the non-edges. The columns of Ψ correspond to the $\overline{m}$ variables in X, and the row for (i,j) represents the equation $(AX)_{i,j} = 0$. Therefore, a matrix has the SAP if and only if the corresponding SAP matrix is full-rank.

The rows of Ψ can be partitioned into n blocks, each having n elements. The k-th block are those rows indexed by (i,k) for $1 \le i \le n$. Let $\mathbf{v}_j$ be the j-th column of A. For the submatrix of Ψ induced by the rows in the k-th block, the $\{j,h\}$ column is $\mathbf{v}_j$ if $k \in \{j,h\}$ and $k = h$, and is a zero vector otherwise. Equivalently, on the $\{i,j\}$ column of Ψ, the i-th block is $\mathbf{v}_j$, the j-th block is $\mathbf{v}_i$, while other blocks are zero vectors.

Example 3.1.3. Let $G = P_4$ be the path on four vertices, with the vertices labeled by $\{1,2,3,4\}$ in the path order. Consider a matrix $A \in \mathcal{S}(G)$ and the matrix X with three variables, as shown below.

$$AX = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & x_{\{1,3\}} & x_{\{1,4\}} \\ 0 & 0 & 0 & x_{\{2,4\}} \\ x_{\{1,3\}} & 0 & 0 & 0 \\ x_{\{1,4\}} & x_{\{2,4\}} & 0 & 0 \end{bmatrix} \end{matrix}$$

The SAP matrix of A with respect to the order $(\{1,3\},\{1,4\},\{2,4\})$ is a matrix Ψ representing the linear system for $AX = O$ with three variables $x_{\{1,3\}}, x_{\{1,4\}}, x_{\{2,4\}}$. For convenience, write $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$, where $\mathbf{v}_j$ is the j-th column vector of A. Now

$AX = O$ means

$$\sum_{j \notin N_G[k]} x_{\{j,k\}} \mathbf{v}_j = \mathbf{0} \text{ for each } k \in V(G).$$

Thus,

$$\Psi = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \end{array} \\ \begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \end{array} = \begin{array}{c} \\ (1,1) \\ (2,1) \\ (3,1) \\ (4,1) \\ (1,2) \\ (2,2) \\ (3,2) \\ (4,2) \\ (1,3) \\ (2,3) \\ (3,3) \\ (4,3) \\ (1,4) \\ (2,4) \\ (3,4) \\ (4,4) \end{array} \begin{array}{c} \begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.$$

3.1.2 Zero forcing parameters

On a graph G, the conventional *zero forcing game* (ZFG) is a color-change game such that each vertex is colored blue or white initially, and then the *color change rule* (CCR) is applied repeatedly. If starting with an initial blue set $B \subseteq V(G)$ and every vertex turns blue eventually, this set B is called a *zero forcing set* (ZFS). The zero forcing number is defined as the minimum cardinality of a ZFS.

Different types of zero forcing numbers are discussed in the literature (e.g., see [3, 4, 12]). Most of them serve as upper bounds of different types of maximum nullities. Here we consider three types of the zero forcing numbers Z, Z_ℓ, Z_+ with the corresponding color change rules:

- (CCR-Z) If i is a blue vertex and j is the only white neighbor of i, then j may turn blue.
- (CCR-Z_ℓ) CCR-Z can be used to perform a force. Or if i is a white vertex without white neighbors and i is not isolated, then i may turn blue.
- (CCR-Z_+) Let B be the set of blue vertices at some stage and W the vertices of a component of $G - B$. CCR-Z is applied to $G[B \cup W]$ with blue vertices B.

When a zero forcing game is mentioned, it is equipped with a color change rule, and we use $i \to j$ to denote a corresponding force (i.e. i forcing j to become blue). Note that for CCR-Z_ℓ, it is possible to have $i \to i$. Also, at the same stage, the color change rule might be able to apply to different i and j (or W for CCR-Z_+), so the player has the choice to decide where to apply the rule, though the final coloring where no more color change rules can be applied is always the same.

It is known [2–4] that $M(G) \le Z(G)$, $M_+(G) \le Z_+(G)$, and $Z_+(G) \le Z_\ell(G) \le Z(G)$. Denote $\mathcal{S}_\ell(G)$ as those matrices in $\mathcal{S}(G)$ whose i,i-entry is zero if and only if vertex i is an isolated vertex. Then every matrix $A \in \mathcal{S}_\ell(G)$ has nullity at most $Z_\ell(G)$ [13].

All these results rely on Proposition 3.1.4.

Proposition 3.1.4. [2, 3, 13] *Let G be a graph on n vertices. Suppose at some stage B is the set of blue vertices.*

- *If $i \to j$ under CCR-Z, then for any matrix $A \in \mathcal{S}(G)$ with column vectors $\{\mathbf{v}_s\}_{s=1}^n$, $\sum_{s \notin B} x_s \mathbf{v}_s = \mathbf{0}$ implies $x_j = 0$.*

- *If $i \to j$ under CCR-Z_ℓ, then for any matrix $A \in \mathcal{S}_\ell(G)$ with column vectors $\{\mathbf{v}_s\}_{s=1}^n$, $\sum_{s\notin B} x_s\mathbf{v}_s = \mathbf{0}$ implies $x_j = 0$.*

- *If $i \to j$ under CCR-Z_+, then for any matrix $A \in \mathcal{S}_+(G)$ with column vectors $\{\mathbf{v}_s\}_{s=1}^n$, $\sum_{s\notin B} x_s\mathbf{v}_s = \mathbf{0}$ implies $x_j = 0$.*

3.2 SAP zero forcing parameters

In this section, we introduce a new parameter $Z_{\text{SAP}}(G)$ and prove that if $Z_{\text{SAP}}(G) = 0$ then every matrix $A \in \mathcal{S}(G)$ has the SAP, which implies $M(G) = \xi(G)$. We also introduce similar parameters and results for other variants.

First we give two examples illustrating what we called in Definition 3.2.4 the forcing triple and the odd cycle rule.

Example 3.2.1. Consider the graph P_4. Let A be the matrix as in Example 3.1.3 and $\mathbf{v}_j$ its j-th column. In Example 3.1.3, we know the SAP matrix of A can be written as

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \end{array} \\ \begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \end{array}.$$

Since $\mathbf{v}_4$ is the only nonzero vector on the second block-row, $x_{\{2,4\}}$ must be 0 in this linear system. Similarly, $\mathbf{v}_1$ is the only nonzero vector on the third block-row, so $x_{\{1,3\}} = 0$. Provided that $x_{\{1,3\}} = x_{\{2,4\}} = 0$, the structure on the first block-row forces $x_{\{1,4\}} = 0$. Since this argument holds for every matrix in $\mathcal{S}(G)$, every matrix in $\mathcal{S}(G)$ has the SAP.

Example 3.2.2. Let $G = K_{1,3}$. Consider the matrices A and X as

$$A = \begin{bmatrix} d_1 & a_1 & a_2 & a_3 \\ a_1 & d_2 & 0 & 0 \\ a_2 & 0 & d_3 & 0 \\ a_3 & 0 & 0 & d_4 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_{\{2,3\}} & x_{\{2,4\}} \\ 0 & x_{\{2,3\}} & 0 & x_{\{3,4\}} \\ 0 & x_{\{2,4\}} & x_{\{3,4\}} & 0 \end{bmatrix}.$$

Let $\mathbf{v}_j$ be the j-th column of A. Then the SAP matrix of A with respect to the order $(\{2,3\},\{3,4\},\{2,3\})$ can be written as

$$\Psi = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x_{\{2,3\}} & x_{\{3,4\}} & x_{\{2,4\}} \end{array} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{v}_3 & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_2 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_3 & \mathbf{v}_2 \end{bmatrix} \end{array}.$$

Recall that the row with index (i,j) is the i-th row in the j-th block. Thus the submatrix induced by rows $\{(1,2),(1,3),(1,4)\}$ is

$$\begin{bmatrix} a_2 & 0 & a_3 \\ a_1 & a_3 & 0 \\ 0 & a_2 & a_1 \end{bmatrix},$$

whose determinant is always nonzero if $a_1, a_2, a_3 \neq 0$. This means the SAP matrix of A is always full-rank, regardless the choice of $A \in \mathcal{S}(G)$. Hence every matrix $A \in \mathcal{S}(G)$ has the SAP. The reason behind this is because a 3-cycle appears in $\overline{G}$.

As shown in Example 3.2.1 and Example 3.2.2, some graph structures guarantee that every matrix described by the graph has the SAP. This assurance is given by forcing $x_e = 0$ step by step or by the occurrence of some odd cycle inside $\overline{G}$. Utilizing these ideas, we design the *SAP zero forcing game*, where the information $x_e = 0$ is stored by coloring the non-edge e blue.

Different from the conventional zero forcing game, the SAP zero forcing game is coloring "non-edges" to be blue or white, instead of coloring vertices; also, a set of initial blue non-edges is called a zero forcing set if every non-edge turns blue eventually by repeated applications of the given color change rules.

Let G be a graph and $i \in V(G)$. Recall that $N_G(i)$ is the neighborhood of i in G. For B_E a set of edges (2-sets), by considering $\langle B_E \rangle$ as the graph with its edge set B_E on the required vertices, the notation $N_{\langle B_E \rangle}(i)$ denotes the vertices j with $\{i, j\} \in B_E$.

The definition of $Z_{\text{SAP}}(G)$ uses the concept of local games, which we now define.

Definition 3.2.3. Let G be a graph with some non-edges B_E colored blue, and $k \in V(G)$. The *local game* $\phi_Z(G, B_E, k)$ is the conventional zero forcing game on G equipped with CCR-Z and the initial blue set $\phi_k(G, B_E) := N_G[k] \cup N_{\langle B_E \rangle}(k)$. When Z is replaced by another zero forcing rules, such as Z_ℓ or Z_+, the setting remains the same but a different rule is applied.

Definition 3.2.4. For a graph G, the *SAP zero forcing number* $Z_{\text{SAP}}(G)$ is the minimum number of blue non-edges such that every non-edge will become blue by repeated applications of the *color change rule for* Z_{SAP} (CCR-Z_{SAP}):

- Suppose at some stage, B_E is the set of blue non-edges and $\{j, k\}$ is a white non-edge. If $i \to j$ in $\phi_Z(G, B_E, k)$ for some vertex i, then the non-edge $\{j, k\}$ may turn blue. This is denoted as $(k : i \to j)$.
- Let $\overline{G}_W$ be the graph whose edges are the white non-edges. If for some vertex i, $\overline{G}_W[N_G(i)]$ contains a component that is an odd cycle C, then all edges in $E(C)$ may turn blue. This is denoted as $(i \to C)$.

The three vertices i, j, and k in the first rule are called a *forcing triple*; the second rule is called the *odd cycle rule*.

Note that a complete graph $G = K_n$ is considered as having all non-edges blue initially, so $Z_{\mathrm{SAP}}(G) = 0$. The odd cycle rule follows a similar idea from the odd cycle zero forcing number [20].

Lemma 3.2.5. *For any nonzero real numbers $a_1, a_2, \ldots, a_n$ with n odd, a matrix of the form*

$$\begin{bmatrix} a_2 & 0 & \cdots & 0 & a_n \\ a_1 & a_3 & 0 & & 0 \\ 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & a_n & 0 \\ 0 & \cdots & 0 & a_{n-1} & a_1 \end{bmatrix}$$

is nonsingular.

Proof. Let A be a matrix of the described form. When n is odd,

$$\det(A) = 2\prod_{i=1}^{n} a_i,$$

which is nonzero provided that a_i's are all nonzero. Hence A is nonsingular. □

Theorem 3.2.6. *Suppose G is a graph with $Z_{\mathrm{SAP}}(G) = 0$. Then every matrix in $\mathcal{S}(G)$ has the SAP. Therefore, $M(G) = \xi(G)$, $M_+(G) = \nu(G)$, and $M_\mu(G) = \mu(G)$.*

Proof. Let $A = \begin{bmatrix} a_{i,j} \end{bmatrix} \in \mathcal{S}(G)$ with $\mathbf{v}_j$ as the j-th column vector. Pick an order for the set of non-edges, and let Ψ be the SAP matrix for A with respect to the given order. Suppose $\mathbf{x}$ is a vector such that $\Psi\mathbf{x} = \mathbf{0}$. Then $\mathbf{x} = (x_e)_{e\in E(\overline{G})}$ such that the entries of $\mathbf{x}$ are indexed by the non-edges of G in the given order. We relate the SAP zero forcing game to the zero-nonzero pattern of $\mathbf{x}$.

Claim 1: Suppose at some stage, B_E is the set of blue non-edges, and $(k : i \to j)$ is a forcing triple. Then $x_e = 0$ for all $e \in B_E$ implies $x_{\{j,k\}} = 0$.

To establish the claim, recall that the condition $\Psi\mathbf{x} = \mathbf{0}$ on those rows in the k-th block means

$$\sum_{s\notin N_G[k]} x_{\{s,k\}}\mathbf{v}_s = \mathbf{0}.$$

Suppose $x_e = 0$ for all $e \in B_E$. Then this equality reduces to

$$\sum_{s \notin N_G[k] \cup N_{\langle B_E \rangle}(k)} x_{\{s,k\}} \mathbf{v}_s = \mathbf{0}.$$

Since by Definition 3.2.3 the set $\phi_k(G, B_E) = N_G[k] \cup N_{\langle B_E \rangle}(k)$ is exactly the set of initial blue vertices in $\phi_Z(G, B_E, k)$, the force $i \to j$ in $\phi_Z(G, B_E, k)$ implies $x_{\{j,k\}} = 0$ by Proposition 3.1.4.

Claim 2: Suppose at some stage, B_E is the set of blue non-edges, and $(i \to C)$ is applied by the odd cycle rule. Then $x_e = 0$ for all $e \in B_E$ implies $x_e = 0$ for every $e \in E(C)$.

To establish the claim, let $\overline{G}_W$ be the graph whose edges are the white non-edges at this stage. Since $(i \to C)$ is applied by the odd cycle rule, C is a component in $\overline{G}_W[N_G(i)]$ and $|V(C)| = d$ is an odd number. Following the cycle order, write the vertices in $V(C)$ as $\{k_s\}_{s=1}^d$, and $e_s = \{k_s, k_{s+1}\}$, with the index taken modulo d.

Denote $U = \{(i, k_s)\}_{s=1}^d$, $W_1 = \{e_s\}_{s=1}^d$, and W_2 as those white non-edges not in W_1. Now $\{B_E, W_1, W_2\}$ forms a partition of $E(\overline{G})$. We have no control about $\Psi[U, B_E]$, but will show $\Psi[U, W_1]$ is always nonsingular and $\Psi[U, W_2] = O$. Consequently, $x_e = 0$ for all $e \in B_E$ implies $x_e = 0$ for every non-edge $e \in W_1 = E(C)$.

For each $(i, k_s) \in U$, $\Psi_{(i,k_s),e_{s-1}} = a_{i,k_{s-1}}$ and $\Psi_{(i,k_s),e_s} = a_{i,k_{s+1}}$, while both of them are nonzero; at the same time, $\Psi_{(i,k_s),e} = 0$ for all $e \in W_1$ other than e_{s-1} and e_s, since e is not incident to k_s. Therefore, $\Psi[U, W_1]$ is of the form described in Lemma 2.5, and it must be nonsingular.

On the other hand, consider a non-edge $\{j, h\} \in W_2$ and $(i, k_s) \in U$. If $k_s \notin \{j, h\}$, then $\Psi_{(i,k_s),\{j,h\}} = 0$. If $k_s \in \{j, h\}$, say $k_s = h$, then $j \notin N_G(i)$ (for otherwise k_s has degree at least 3 in $\overline{G}_W[N_G(i)]$ and the component containing k_s cannot be an odd cycle); this means $\{i, j\} \notin E(G)$ and $\Psi_{(i,k_s),\{j,h\}} = a_{i,j} = 0$. Therefore, $\Psi[U, W_2] = O$.

By the claims, $Z_{\text{SAP}}(G) = 0$ means all of the x_e will be forced to zero, so $\mathbf{x} = \mathbf{0}$ is the only vector in the right kernel of Ψ. This means Ψ is full-rank.

Since the argument works for every matrix $A \in \mathcal{S}(G)$, $Z_{\text{SAP}}(G) = 0$ implies every matrix $A \in \mathcal{S}(G)$ has the SAP. Consequently, $M(G) = \xi(G)$, $M_+(G) = \nu(G)$, and $M_\mu(G) = \mu(G)$. □

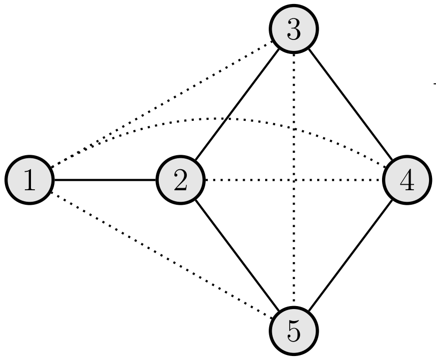
Remark 3.2.7. With and without the restriction of having the SAP, the inertia sets that can be achieved by matrices in $\mathcal{S}(G)$ are considered in the literature (e.g., see [1,6]). With the help of Theorem 3.2.6, if $Z_{\text{SAP}}(G) = 0$, then these two inertia sets are the same.

Corollary 3.2.8. *If G has no isolated vertices and $\overline{G}$ is a forest, then $Z_{\text{SAP}}(G) = 0$ and every matrix in $\mathcal{S}(G)$ has the SAP.*

Proof. Suppose at some stage $\overline{G}_W$ is the graph whose edges are the white non-edges. Since $\overline{G}$ is a forest, $\overline{G}_W$ always has a leaf k, unless $\overline{G}_W$ contains no edges. Let j be the only neighbor of k in $\overline{G}_W$, and let i be one of the neighbors of j in G. Since G has no isolated vertices, i always exists. Thus, in the local game $\phi_Z(G, E(\overline{G}) \setminus E(\overline{G}_W), k)$, every vertex is blue except j, so $i \to j$. Therefore, $(k : i \to j)$ can be applied and $\{j, k\}$ turns blue. Continuing this process, all non-edges become blue, so $Z_{\text{SAP}}(G) = 0$. □

Note that the condition that G has no isolated vertices is crucial for Corollary 3.2.8. For example, $Z_{\text{SAP}}(\overline{K_{1,n}}) > 0$ when $n \geq 1$. In fact, $Z_{\text{SAP}}(G) = 0$ does not happen only when $\overline{G}$ is a forest. Example 3.2.9 gives a graph G such that $\overline{G}$ is not a forest and $Z_{\text{SAP}}(G) = 0$. We will see in Table 3.1 that there are a considerable number of graphs having the property $Z_{\text{SAP}}(G) = 0$.

Example 3.2.9. Let G be the graph shown in Figure 3.2. Following the steps listed in Figure 3.2, every non-edge turns blue, so $Z_{\text{SAP}}(G) = 0$. Observe that at the beginning, the graph $\overline{G}_W$ of white non-edges is the same as $\overline{G}$, and $\overline{G}_W[N_G(2)]$ is a 3-cycle C, so one can also use the odd cycle rule to perform $(2 \to C)$. This will accelerate the process but not change the result. By Theorem 3.2.6, every matrix $A \in \mathcal{S}(G)$ has the SAP, so $\xi(G) = M(G)$. Since the number of vertices is no more than 7, $M(G) = Z(G) = 2$ and thus $\xi(G) = 2$.



Step	Forcing triple	Forced non-edge
1	$(2:3\to 4)$	$\{2,4\}$
2	$(4:2\to 1)$	$\{4,1\}$
3	$(5:4\to 3)$	$\{5,3\}$
4	$(3:2\to 1)$	$\{3,1\}$
5	$(5:2\to 1)$	$\{5,1\}$

Figure 3.2 The graph G for Example 3.2.9 and the forcing process

Similar to Corollary 3.2.8, Remark 3.2.10 also provides some intuition of the SAP zero forcing process.

Remark 3.2.10. Suppose at some stage B_E is the set of blue non-edges on a graph G. Let $\overline{G}_W$ be the graph whose edges are the white non-edges. If for some vertex i, the induced subgraph $\overline{G}_W[N_G(i)]$ has a leaf k with its only neighbor j in $\overline{G}_W[N_G(i)]$, then $(k:i\to j)$ can be applied on G, because in $\phi_Z(G,B_E,k)$ every vertex in $N_G(i)$ is blue except j.

This means whenever $\overline{G}_W[N_G(i)]$ contains a component that is a tree, every non-edge in this tree can turn blue inductively by forcing triples. Consequently, if at some stage $(i\to C)$ can be applied but some non-edges from C turns blue because of some forcing triples or other odd cycle rules, all edges in $E(C)$ can still turn blue by forcing triples, but not by $(i\to C)$.

Corollary 3.2.11. *Let G be any graph with diameter 2 and maximum degree at most 3. Then $Z_{\text{SAP}}(G)=0$. In particular, when G is the Petersen graph, $Z_{\text{SAP}}(G)=0$, so $\xi(G)=M(G)=5$.*

Proof. For every white non-edge $\{j,k\}$, there is at least one common neighbor i of j and k, since the diameter is 2. By the assumption, $\deg_G(i)\leq 3$. Since i has at least two neighbors, $\deg_G(i) \geq 2$. If $\deg_G(i) = 2$, then $(k : i \to j)$. Suppose $\deg_G(i) = 3$. On the set $N_G(i)$, the white non-edges can form P_2, P_3, or C_3. In the case of P_2 and P_3, all non-edges in $N_G(i)$ turn blue by the argument in Remark 3.2.10. If it is C_3, then apply the odd cycle rule $(i \to C)$. Since this argument works for every white non-edge, all non-edges can turn blue. Hence $Z_{\mathrm{SAP}}(G) = 0$.

Let G be the Petersen graph. Then G is a 3-regular graph with diameter 2. Thus, $Z_{\mathrm{SAP}}(G) = 0$, and $\xi(G) = M(G)$ by Theorem 3.2.6. It is known [2] that $M(G) = 5$. □

In [5], it is asked if $\xi(G) \leq \xi(G - v) + 1$ for every graphs G and every vertex v of G. Theorem 3.2.6 answers this question in positive for a large number of graph-vertex pairs.

Corollary 3.2.12. *Let G be a graph and $v \in V(G)$. Suppose $Z_{\mathrm{SAP}}(G - v) = 0$. Then $\xi(G) \leq \xi(G - v) + 1$.*

Proof. Since $Z_{\mathrm{SAP}}(G - v) = 0$, $\xi(G - v) = M(G - v)$ by Theorem 3.2.6. Therefore,

$$\xi(G) \leq M(G) \leq M(G - v) + 1 = \xi(G - v) + 1,$$

where the inequality $M(G) \leq M(G - v) + 1$ is given in [11]. □

Example 3.2.13. Let G be one of the tetrahedron K_4, cube Q_3, octahedron G_8, dodecahedron G_{12}, or icosahedron G_{20}. Then, $Z_{\mathrm{SAP}}(G) = 0$. This is trivial for tetrahedron, since it is a complete graph. The complement of an octahedron is three disjoint edges, which is a forest, so $Z_{\mathrm{SAP}}(G) = 0$. For the other three graphs, pick one vertex i and look at its neighborhood $N_G(i)$. The induced subgraph of $\overline{G}$ on $N_G(i)$ is either a 3-cycle or a 5-cycle. Thus the odd cycle rule or the argument in Remark 3.2.10 could be applied, and every non-edge in $N_G(i)$ turns blue. After doing this to every vertex, by picking one vertex and look at its local game, all white non-edges incident to this vertex will turn blue. Therefore, $\xi(G) = M(G)$.

It is known [16] that $M(K_4) = 3$ and $M(Q_3) = 4$. Since the octahedron graph is strongly regular, in [2] it shows $4 \leq M(G_8)$; together with the fact $Z(G_8) \leq 4$, we know

$M(G_8) = 4$. For G_{12} and G_{20}, the zero forcing numbers can be computed through the computer program and both equal to 6, but the maximum nullity is not yet known.

Definition 3.2.14. Let G be a graph with some non-edges B_E colored blue. The color change rule for Z^+_{SAP} (CCR-Z^+_{SAP}) is the following:

- Let $\{j,k\}$ be a non-edge. If $i \to j$ in $\phi_{Z_+}(G, B_E, k)$ for some vertex i, then the non-edge $\{j,k\}$ may turn to blue. This is denoted as $(k : i \to j)$.
- The odd cycle rule can be used to perform a force.

Similarly, the color change rule of Z^ℓ_{SAP} (CCR-Z^ℓ_{SAP}) is defined through the local game $\phi_{Z_\ell}(G, B_E, i)$. As usual, $Z^+_{\text{SAP}}(G)$ (respectively, Z^ℓ_{SAP}) is the minimum number of blue non-edges such that every non-edge will become blue by repeated applications of CCR-Z^+_{SAP} (respectively, CCR-Z^ℓ_{SAP}).

Observation 3.2.15. For any graph G, $Z^+_{\text{SAP}}(G) \le Z^\ell_{\text{SAP}}(G) \le Z_{\text{SAP}}(G)$.

By a proof analogous to that of Theorem 3.2.6, we can establish Theorem 3.2.16. Observe that $Z^\ell_{\text{SAP}}(G) = 0$ implies $Z^+_{\text{SAP}}(G) = 0$.

Theorem 3.2.16. *Let G be a graph. If $Z^\ell_{\text{SAP}}(G) = 0$, then every matrix in $\mathcal{S}_\ell(G)$ has the SAP. If $Z^+_{\text{SAP}}(G) = 0$, then every matrix in $\mathcal{S}_+(G)$ has the SAP. Therefore, if $Z^+_{\text{SAP}}(G) = 0$, then $M_+(G) = \nu(G)$.*

Corollary 3.2.17. *Suppose G is a graph with $Z^+_{\text{SAP}}(G) = 0$. Then $\xi(G) \ge M_+(G)$.*

Example 3.2.18. Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete multi-partite graph with $n_1 \ge n_2 \ge \cdots \ge n_p$ and $p \ge 2$. Denote $n = \sum_{t=1}^{p} n_t$. Then $Z^\ell_{\text{SAP}}(G) = Z^+_{\text{SAP}}(G) = 0$, so $\nu(G) = M_+(G) = n - n_1$ [12]. On the other hand, if $n_1 \ge 4$, then $Z_{\text{SAP}}(G) > 0$, since none of the non-edges in this part can turn blue.

Example 3.2.19. If T is a tree, then $Z^+_{\text{SAP}}(T) = 0$. However, not every tree T has $Z^\ell_{\text{SAP}}(T) = 0$. For example, let G be the graph obtained from $K_{1,4}$ by attaching four leaves to the four existing leaves. In this graph, only the non-edges incident to the center vertex can turn blue by CCR-Z^{ℓ}_{SAP}, so $Z^{\ell}_{\text{SAP}}(G) > 0$.

3.2.1 Graph join

Since the SAP zero forcing process uses a propagation on non-edges, it is interesting to consider $Z_{\text{SAP}}(G)$ if $\overline{G}$ has two or more components; that is, G is a join of two or more graphs.

Proposition 3.2.20. *Let G and H be two graphs. Then*

$$Z_{\text{SAP}}(G \vee H) = Z_{\text{SAP}}(G \vee K_1) + Z_{\text{SAP}}(H \vee K_1).$$

Proof. Let v be the vertex corresponding to the K_1 in $G \vee K_1$. Denote $E_1 = E(\overline{G})$ and $E_2 = E(\overline{H})$. Consider the mapping $\pi : V(G \vee H) \to V(G \vee K_1)$ such that $\pi(i) = i$ if $i \in V(G)$ and $\pi(i) = v$ if $i \in V(H)$. Fix a vertex $u \in V(H)$, consider the mapping $\pi^{-1} : V(G \vee K_1) \to V(G \vee H)$ such that $\pi^{-1}(i) = i$ if $i \in V(G)$ and $\pi^{-1}(v) = u$.

Suppose at some stage B_E is the set of blue non-edges in $G \vee H$, and $B_E \cap E_1$ and $B_E \cap E_2$ are the sets of blue non-edges in $G \vee K_1$ and $H \vee K_1$ respectively. Let $e = \{j, k\} \in E_1$. If $(k : i \to j)$ happens in $G \vee H$, then $(k : \pi(i) \to j)$ can be applied in $G \vee K_1$; if $(k : i \to j)$ happens in $G \vee K_1$, then $(k : \pi^{-1}(i) \to j)$ can be applied in $G \vee H$. Also, if e is in some cycle C and $(i \to C)$ happens in either $G \vee H$ or $G \vee K_1$, then by the definition of the odd cycle rule C must totally fall in $V(G)$. If $(i \to C)$ in $G \vee H$, then $(\pi(i) \to C)$ in $G \vee K_1$; if $(i \to C)$ in $G \vee K_1$, then $(\pi^{-1}(i) \to C)$ in $G \vee H$. Similarly, all these correspondences work when $e \in E_2$.

Therefore, we can conclude that B_E is a ZFS-Z_{SAP} in $G \vee H$ if and only if $B_E \cap E_1$ and $B_E \cap E_2$ are ZFS-Z_{SAP} in $G \vee K_1$ and $H \vee K_1$ respectively. □

Example 3.2.21. The value of $Z_{\text{SAP}}(G \vee K_1)$ and the value of $Z_{\text{SAP}}(G)$ can vary a lot. For example, when $G = \overline{K_n}$, we will show that $Z_{\text{SAP}}(\overline{K_n}) = \binom{n}{2}$ and $Z_{\text{SAP}}(\overline{K_n} \vee K_1) = Z_{\text{SAP}}(K_{1,n}) = \binom{n-1}{2} - 1$ when $n \geq 3$.

Since there are no edges in $\overline{K_n}$, no vertices can make a force in any local game, and odd cycle rules cannot be applied, either. This means $Z_{\text{SAP}}(\overline{K_n}) = \binom{n}{2}$.

For $K_{1,n}$, color some edges B_E of $\overline{K_{1,n}}$ blue so that the set of white non-edges forms a 3-cycle with $n-3$ leaves attaching to a vertex of the 3-cycle. Then B_E is a ZFS-Z_{SAP} for $K_{1,n}$, since the $n-3$ leaves can turn blue by forcing triples, and then the 3-cycle can turn blue by the odd cycle rule. Therefore, $Z_{\text{SAP}}(K_{1,n}) \leq \binom{n-1}{2} - 1$.

The inequality $Z_{\text{SAP}}(K_{1,3}) \leq \binom{3-1}{2} - 1 = 0$ implies $Z_{\text{SAP}}(K_{1,3}) = 0$, so we may assume $n \geq 4$. Suppose B_E is a ZFS-Z_{SAP} of $K_{1,n}$ with $|B_E| = \binom{n-1}{2} - 2$. Let $\overline{G}_W$ be the graph whose edges are the white non-edges. Then $|E(\overline{G}_W)| = n+1$. Obtain a subgraph H of $\overline{G}_W$ by deleting leaves and isolated vertices repeatedly until there is no leaves or isolated vertices left. By the choice of H, it is either $|V(H)| = 0$ or H has the minimum degree at least two. Since deleting a leaf removes an edge and a vertex, $|V(H)| + 1 \leq |E(H)|$, implying $|E(H)| \neq 0$ and $|V(H)| \neq 0$. Now H is a graph with minimum degree at least two and $|V(H)| + 1 \leq |E(H)|$; therefore, H must contain a component that is not a cycle (so in particular not an odd cycle). Let $\{j,k\}$ be an edge in this component. If $(k : i \to j)$ force $\{j,k\}$ to turn blue for some i, then i must be the center vertex of $K_{1,n}$. However, in $\phi_Z(G, B_E, k)$, vertex i has at least two white neighbors, because k has degree at least two in H. Therefore, no edges in this component can turn blue by either a forcing triple or an odd cycle rule, a contradiction. Hence $Z_{\text{SAP}}(K_{1,n}) = \binom{n-1}{2} - 1$.

Proposition 3.2.22. *For any graph G, $Z_{\text{SAP}}(G \vee K_1) \leq Z_{\text{SAP}}(G)$. If G contains no isolated vertices, then $Z_{\text{SAP}}(G \vee K_1) = Z_{\text{SAP}}(G)$.*

Proof. Every ZFS-Z_{SAP} for G is a ZFS-Z_{SAP} for $G \vee K_1$, so $Z_{\text{SAP}}(G \vee K_1) \leq Z_{\text{SAP}}(G)$.

Now consider the case that G has no isolated vertices. Suppose at some stage B_E is the set of blue non-edges for both $G \vee K_1$ and G. We claim that if a non-edge $\{j,k\} \in E(\overline{G})$

turns blue in $G \vee K_1$, then it can also turn blue in G.

Label the vertex in $V(K_1)$ as v. If $(k : i \to j)$ in $G \vee K_1$ with $i \neq v$, then it is also a forcing triple in G. Suppose $(k : v \to j)$ happens in $G \vee K_1$. Then it must be the case when j is the only white vertex in $\phi_Z(G \vee K_1, B_E, k)$, since v is a vertex that is adjacent to every vertex and it cannot make a force unless every vertex except j is already blue. Since j is not an isolated vertex, it has a neighbor i' in $V(G)$. Now $(k : i' \to j)$ can force $\{j, k\}$ to turn blue, since j is also the only white vertex in $\phi_Z(G, B_E, k)$.

On the other hand, if $(i \to C)$ happens in $G \vee K_1$ with $i \neq v$, then it can also happen in G. Suppose $(v \to C)$. Then every vertex in C is incident to exactly two white non-edges by the odd cycle rule, since v is adjacent to every vertex. Label the vertices of C by $\{k_s\}_{s=1}^{d}$ in the cycle order, with the index taken modulo d. In the local game $\phi_Z(G, B_E, k_2)$, there are only two white vertices, namely k_1 and k_3. Since G has no isolated vertices, k_1 has a neighbor i' in $V(G)$. If i' is not adjacent to k_3, then $(k_2 : i' \to k_1)$ can be applied and then the argument in Remark 3.2.10 can force all edges in $E(C)$ to turn blue. Therefore, we may assume i' is adjacent to k_3. By applying the same argument to k_4, we know i' is also adjacent to k_5. Inductively, i' is adjacent to all vertices in C, since C is an odd cycle. Therefore, $(i' \to C)$ can happen in G.

In conclusion, $Z_{\text{SAP}}(G \vee K_1) = Z_{\text{SAP}}(G)$. □

Proposition 3.2.23. *Let G be a graph. Then $Z_{\text{SAP}}(G \vee K_1) = 0$ if and only if one of the following holds:*

- *G has no isolated vertices and $Z_{\text{SAP}}(G) = 0$.*
- *$G = K_1$ or G is a disjoint union of a connected graph H and an isolated vertex such that $Z_{\text{SAP}}(H) = 0$.*
- *$G = \overline{K_3}$.*

Proof. Let v be the vertex in $V(K_1) \subseteq V(G \vee K_1)$. In the case that G has no isolated vertices, $Z_{\text{SAP}}(G \vee K_1) = 0$ if and only if $Z_{\text{SAP}}(G) = 0$ by Proposition 3.2.22. If $G = K_1$, then $Z_{\mathrm{SAP}}(K_2) = 0$. If $G = \overline{K_3}$, then $Z_{\mathrm{SAP}}(K_{1,3}) = 0$. Finally, suppose G is a disjoint union of a connected graph H and an isolated vertex w such that $Z_{\mathrm{SAP}}(H) = 0$. Then every forcing triple and every odd cycle rule in H can work in $G \vee K_1$, so all non-edges of $G \vee K_1$ that are in the part of H can turn blue. After that, $(k : v \to w)$ takes action in $G \vee K_1$ for every $k \in V(H)$. Thus, every non-edge in $G \vee K_1$ is blue.

For the converse statement, suppose $Z_{\mathrm{SAP}}(G \vee K_1) = 0$ and no initial blue non-edge is given for $G \vee K_1$. Suppose G has p components with vertex sets $V_1, V_2, \ldots, V_p$. Call a non-edge with two endpoints in different components in G as a crossing non-edge. We claim that if $p \geq 3$, then no crossing non-edge can turn blue in $G \vee K_1$ by any forcing triples. Let $\{j, k\}$ be a crossing non-edge. Without loss of generality, let $k \in V_1$ and $j \in V_2$. Suppose at some stage B_E is the set of blue non-edges and none of the crossing non-edges is blue. In the local game $\phi_Z(G \vee K_1, B_E, k)$, all blue vertices are contained in $V_1 \cup \{v\}$, since all the crossing non-edges are white. If $(k : i \to j)$ happens in $G \vee K_1$, it must be the case that $i = v$, since v is the only blue neighbor of j in $\phi_Z(G \vee K_1, B_E, k)$. Pick a vertex $u \in V_3$. Since both j and u are white neighbors of v in $\phi_Z(G \vee K_1, B_E, k)$, it is impossible that $(k : i \to j)$ is a forcing triple. In conclusion, if $Z_{\mathrm{SAP}}(G \vee K_1) = 0$ and G contains at least three components, then the odd cycle rule must be applied to the crossing non-edges. Therefore, G must be $\overline{K_3}$ in this case.

If G has only one component, then G contains no isolated vertices, unless $G = K_1$. Otherwise assume G has an isolated vertex and has exactly two components. Then G must be a disjoint union of a connected graph H and an isolated vertex w. Now we build a sequence of forces for H according to the forces in $G \vee K_1$. Suppose $(k : i \to j)$ happens in $G \vee K_1$ with $j, k \in V(H)$. If $i \in V(H)$, then $(k : i \to j)$ also works in H. If $i \notin V(H)$, then it must be $(k : v \to j)$. But v is adjacent to every vertex, so in $\phi_Z(G, B_E, k)$ every vertex except j must be blue. Since H is connected, there must be a vertex i' that is adjacent to j. Thus, $(k : i' \to j)$ can force $\{j, k\}$ to turn blue.

Suppose $(i \to C)$ for some i and odd cycle C. If $i \in V(H)$, then $V(C) \subset V(H)$ and

$(i \to C)$ can be applied in H. Since i cannot be w, we assume $i = v$. If $w \in V(C)$, then $C - w$ forms a path and all edges on this path can turn blue in H, by the argument of Corollary 3.2.8.

Finally, we claim that $(v \to C)$ cannot happen in $G \vee K_1$ when $V(C) \subseteq V(H)$. For the purpose of obtaining a contradiction, suppose at some stage B_E is the set of blue non-edges and $(v \to C)$ happens. Let $\overline{G}_W$ be the graph whose edges are the white non-edges. Write $V(C) = \{k_s\}_{s=1}^{d}$ in the cycle order, with the index taken modulo d. Since C is a component in $\overline{G}_W[N_{G\vee K_1}(v)]$, every non-edge $\{k_s, w\}$ with $k_s \in V(C)$ is blue at this stage. When all non-edges $\{k_s, w\}$ were all white, no odd cycle rule can force any of them to turn blue, since w is incident to at least $d \geq 3$ white non-edges. Without loss of generality, assume that $\{k_1, w\}$ is the first non-edge to turn blue among $\{k_s, w\}$ with $k_s \in V(C)$. Only a forcing triple can possibly force $\{k_s, w\}$ to turn blue, and it must be $(k_1 : v \to w)$ or $(w : v \to k_1)$. Suppose this happened at the stage where B_{E_0} was the set of blue non-edges. It cannot be $(k_1 : v \to w)$ because v has at least three white neighbors k_2, k_d, and w in $\phi_Z(G \vee K_1, B_{E_0}, k_1)$; meanwhile, it cannot be $(w : v \to k_1)$, since v has at least $d \geq 3$ white neighbors in $\phi_Z(G \vee K_1, B_{E_0}, w)$. This yields a contradiction.

In conclusion, every possible force in $G \vee K_1$ corresponds to a force in H. Therefore, if $Z_{\text{SAP}}(G \vee K_1) = 0$, then $Z_{\text{SAP}}(H) = 0$. □

3.2.2 Computational results for small graphs

Table 3.1 shows the proportions of graphs that have certain parameters equal to 0, over all connected graphs with a fixed number of vertices. Graphs are not labeled and isomorphic graphs are considered as the same. The computation is done by *Sage* and the code can be found in [19].

In Section 3.4, we apply these results to help compute the value of $\xi(G)$ when $|V(G)| \leq 7$.

Table 3.1 The proportion of graphs that satisfies $\zeta(G) = 0$, over all connected graphs on n vertices

n	$Z_{\text{SAP}} = 0$	$Z^{\ell}_{\text{SAP}} = 0$	$Z^{+}_{\text{SAP}} = 0$
1	1.0	1.0	1.0
2	1.0	1.0	1.0
3	1.0	1.0	1.0
4	1.0	1.0	1.0
5	0.86	0.95	0.95
6	0.79	0.92	0.92
7	0.74	0.89	0.89
8	0.73	0.88	0.88
9	0.76	0.89	0.89
10	0.79	0.90	0.91

3.3 A vertex cover version of the SAP zero forcing game

As Example 3.2.21 points out, for a connected graph G on n vertices, the value of $Z_{\text{SAP}}(G)$ can be much higher than n. This section considers a vertex cover version of the SAP zero forcing game. That is, if B is a set of vertices, then consider the *complementary closure* $\overline{cl}(B)$ as all those non-edges that are incident to any vertex in B. Now instead of picking some non-edges as blue at the beginning, we pick a set of vertices B, and color the set $\overline{cl}(B)$ blue initially.

Following this idea, a new parameter $Z_{\text{vc}}(G)$ is defined with $0 \le Z_{\text{vc}}(G) \le n$, and Theorem 3.3.2 shows that $M(G) - Z_{\text{vc}}(G) \le \xi(G)$.

Definition 3.3.1. For a graph G, the parameter $Z_{\text{vc}}(G)$ is the minimum number of vertices B such that by coloring $\overline{cl}(B)$ blue, every non-edge will become blue by repeated applications of CCR-Z_{SAP} with the restriction

- $(k : i \to j)$ cannot perform a force if $i \in B$ and $\{i, k\} \in E(\overline{G})$.

A set $B \subseteq V(G)$ with this property is called a Z_{vc} zero forcing set.

Theorem 3.3.2. *Let G be a graph. Then*

$$M(G) - Z_{\mathrm{vc}}(G) \le \xi(G).$$

Proof. For given G and $A = \begin{bmatrix} a_{i,j} \end{bmatrix} \in \mathcal{S}(G)$, let $d = Z_{\mathrm{vc}}(G)$ and $\overline{m} = |E(\overline{G})|$. Pick an order for the set of non-edges, and let Ψ be the SAP matrix for A with respect to the given order. Let B be a ZFS-Z_{vc} with $|B| = d$. We will show that we can perturb the diagonal entries of A corresponding to B such that the new matrix has the SAP.

Let $W = E(\overline{G}) - \overline{cl}(B)$ be the initial white non-edges. Since B is a ZFS-Z_{vc}, every non-edge in W is forced to turn blue at some stage. Say at stage t, W_t is the set of white non-edges that are forced to turn blue. The set W_t can be one non-edge, or the edges of an odd cycle; thus, $\{W_t\}_{t=1}^{s}$ forms a partition of W, where s is the number of stages it takes to make all non-edges turn blue. Define U_t as follows: If W_t is a non-edge forced to turn blue by the forcing triple $(k : i \to j)$, then $U_t = \{(i,k)\}$; if W_t is a cycle forced by the odd cycle rule $(i \to C)$, then $U_t = \{(i,v)\}_{v \in V(C)}$. Let $U = \bigcup_{t=1}^{s} U_t$.

We first show that $\{U_t\}_{t=1}^{s}$ are mutually disjoint. Let $(i,k) \in U_{t_0}$ at some stage t_0. Suppose $(k : i \to j)$ happens at stage t_0. Right before the force, there must be exactly one white non-edge connecting k and $N_G(i)$, namely $\{j,k\}$, by CCR-Z. After the force, all white non-edges connecting k and $N_G(i)$ turn blue. Suppose $(i \to C)$ happens instead for some odd cycle C. Right before the force, there are exactly two white non-edges connecting k and $N_G(i)$, namely the two edges incident to k in C. After the force, all such white non-edges turn blue already. Therefore, (i,k) can appear in only one stage, and $\{U_t\}_{t=1}^{s}$ are mutually disjoint.

Next we show that $\Psi[U,W]$ is nonsingular. The proof of Theorem 3.2.6 shows that if W_{t_0} is given by the odd cycle rule for some step t_0, then $\Psi[U_{t_0}, W_{t_0}]$ is nonsingular and $\Psi[U_{t_0}, \bigcup_{t=t_0+1}^{s} W_t] = O$. We will see that the same property is also true when W_{t_0} a single non-edge. Suppose at stage t_0, the set of blue non-edges is B_E and $(k : i \to j)$ is applied. Thus, $U_{t_0} = \{(i,k)\}$ and $W_{t_0} = \{\{j,k\}\}$. By Definition 3.1.1,

$$\Psi[U_{t_0}, W_{t_0}] = \begin{bmatrix} \Psi_{(i,k),\{j,k\}} \end{bmatrix} = \begin{bmatrix} a_{i,j} \end{bmatrix},$$

which is nonsingular, since $\{i,j\}$ is an edge. For any white non-edge e that is not incident to k, $\Psi_{(i,k),e} = 0$. If $e = \{j',k\}$ is a white non-edge for some $j' \neq j$, then j' is not a neighbor of i, for otherwise i has two white neighbors in $\phi_Z(G, B_E, k)$; therefore, $\Psi_{(i,k),e} = a_{i,j'} = 0$. By column/row permutations according to $\{W_t\}_{t=1}^d$ and $\{U_t\}_{t=1}^d$ respectively, $\Psi[U,W]$ becomes a lower triangular block matrix, with every diagonal block nonsingular. Hence $\Psi[U,W]$ is nonsingular.

Now give the non-edges in $\overline{cl}(B)$ an order. Following the order, for each non-edge $\{i,j\}$ in $\overline{cl}(B)$, put either (i,j) or (j,i) into another ordered set U_B. Since $\Psi_{(i,j),\{i,j\}} = a_{i,i}$, the diagonal entries of $\Psi[U_B, \overline{cl}(B)]$ are controlled by $a_{i,i}$ for some $i \in B$.

Consider the matrix

$$\Psi[U \cup U_B, W \cup \overline{cl}(B)] = \begin{bmatrix} \Psi[U,W] & \Psi[U,\overline{cl}(B)] \\ \Psi[U_B,W] & \Psi[U_B,\overline{cl}(B)] \end{bmatrix}.$$

We claim that those entry $a_{i,i}$ with $i \in B$ only appear on the diagonal of $\Psi[U_B, \overline{cl}(B)]$. For each $i \in B$, the only possible occurrence of $a_{i,i}$ is in the case $\Psi_{(i,k),\{i,k\}} = a_{i,i}$ for some vertex k and non-edge $\{i,k\} \in E(\overline{G})$. If $i \in B$ and $\{i,k\} \in E(\overline{G})$, then $\{i,k\} \in \overline{cl}(B)$. Therefore, $\Psi[U,W]$ and $\Psi[U_B,W]$ do not have this type of $a_{i,i}$ with $i \in B$ involved. Now it is enough to show $(i,k) \notin U$. Recall that $U = \bigcup_{t=1}^s U_t$. At stage t, if a forcing triple is applied, then $(i,k) \notin U_t$ since $(k : i \to j)$ is forbidden for any j by the definition; if the odd cycle rule is applied, then $(i,k) \notin U_t$ since $\{i,k\} \in E(\overline{G})$. Therefore, $\Psi[U, \overline{cl}(B)]$ contains no such $a_{i,i}$ with $i \in B$, either.

Let D_B be the diagonal matrix indexed by $V(G)$ with the i,i-entry 1 if $i \in B$ and 0 otherwise. Consider the matrix $A + xD_B$. By the discussion above, the SAP matrix of

$A + xD_B$ contains the submatrix

$$\begin{bmatrix} \Psi[U,W] & \Psi[U,\overline{cl}(B)] \\ \Psi[U_B,W] & \Psi[U_B,\overline{cl}(B)] + xI \end{bmatrix}.$$

Since $\Psi[U,W]$ is nonsingular, the submatrix above is nonsingular when x is large enough. This means, by changing $d = |B|$ diagonal entries of A, the corresponding SAP matrix becomes full-rank. Therefore,

$$M(G) - Z_{\text{vc}}(G) \le \text{null}(A + xD_B) \le \xi(G).$$

□

Remark 3.3.3. Theorem 3.3.2 actually proves that if B is a ZFS-Z_{vc}, then every matrix $A \in \mathcal{S}(G)$ attains the SAP by perturbing those diagonal entries corresponding to B.

In classical graph theory, a vertex cover of a graph G is a set of vertices S such that every edge in G is incident to some vertex in S; that is, $G - S$ contains no edges. The *vertex cover number* $\beta(G)$ is defined as the minimum cardinality of a vertex cover in the graph G. Corollary 3.3.4 below shows the relation between $M(G)$, $\xi(G)$, and $\beta(G)$.

Corollary 3.3.4. *Let G be a graph. Then*

$$M(G) - \beta(\overline{G}) \le \xi(G).$$

Proof. Let S be a vertex cover of $\overline{G}$. Then S is a ZFS-Z_{vc}, since every non-edge is blue initially. Therefore, $Z_{\text{vc}}(G) \le \beta(G)$ and the desired inequality comes from Theorem 3.3.2. □

Example 3.3.5. Let $G = K_3 \vee \overline{K_4}$. Then from the data in [9], $M(G) = Z(G) = 5$. Since G is a subgraph of $K_3 \vee P_4$, by minor monotonicity $\xi(G) \le \xi(K_3 \vee P_4) \le Z(K_3 \vee P_4) \le 4$. On the other hand, by picking one of the vertex in $V(K_4)$, it forms a ZFS-Z_{vc}, since the initial white non-edges form a 3-cycle and the odd cycle rule applies. Thus $Z_{\text{vc}}(G) = 1$ and $\xi(G) \ge M(G) - Z_{\text{vc}}(G) = 4$. Therefore, $\xi(G) = 4$.

Notice that G contains a K_4 minor but not a K_5 minor, so we can only say $\xi(G) \geq \xi(K_4) = 3$ by considering K_p minors.

Similarly, we can define $Z^{\ell}_{\text{vc}}(G)$ by changing CCR-Z_{SAP} to CCR-Z^{ℓ}_{SAP}. Then we have Theorem 3.3.6.

Theorem 3.3.6. *Let G be a graph. Then*

$$M_+(G) - Z^{\ell}_{\text{vc}}(G) \leq \nu(G).$$

Remark 3.3.7. The proof of Theorem 3.3.2 relies on the fact $\Psi[U,W]$ is a lower triangular block matrix. This is not always true for Z_+. As a vertex can force two or more white vertices under CCR-Z_+, the sets $\{U_t\}_{t=1}^{s}$ might not be mutually disjoint and it is possible that $|U| < |W|$. Therefore, the same proof does not work for Z_+.

3.4 Values of $\xi(G)$ for small graphs

Analogous to $M(G) \leq Z(G)$, it is shown in [4] that $\xi(G) \leq \lfloor Z \rfloor(G)$, where $\lfloor Z \rfloor(G)$ is defined through a (conventional) zero forcing game with CCR-$\lfloor Z \rfloor$:

- CCR-Z can be used to perform a force. Or if i is blue, i has no white neighbors, and i was not used to make a force yet, then i may pick one white vertex j and force it to turn blue.

By using *Sage* and with the help of Theorem 3.2.6 and Theorem 3.3.2, we will see that $\lfloor Z \rfloor(G)$ agrees with $\xi(G)$ for graphs up to 7 vertices. This result also relies on some other lower bounds. The *Hadwiger number* $\eta(G)$ is defined as the largest p such that G has a K_p minor. Since $\xi(G)$ is minor monotone, it is known [4] that when $\eta(G) = p$

$$\xi(G) \geq \xi(K_p) = p - 1 = \eta(G) - 1.$$

The T_3-family is a family of six graphs [14, Fig. 2.1]. It is known [14] that a graph G contains a minor in the T_3-family if and only if $\xi(G) \geq 3$.

Lemma 3.4.1. *Let G be a connected graph with at most 7 vertices. Then at least one of the following is true:*

- $Z_{\text{SAP}}(G) = 0$, *which implies* $\xi(G) = M(G)$.
- G *is a tree, which implies* $\xi(G) = 2$ *if* G *is not a path, and* $\xi(G) = 1$ *otherwise.*
- $\lfloor Z \rfloor(G) = M(G) - Z_{\text{vc}}(G)$, *which implies* $\xi(G) = \lfloor Z \rfloor(G)$.
- $\lfloor Z \rfloor(G) = \eta(G) - 1$, *which implies* $\xi(G) = \lfloor Z \rfloor(G)$.
- $\lfloor Z \rfloor(G) = 3$ *and* G *contains a* T_3*-family minor, which implies* $\xi(G) = 3$.

Proof. By running a *Sage* program [19], one of the five cases will happen. If $Z_{\text{SAP}}(G) = 0$, then $\xi(G) = M(G)$ by Theorem 3.2.6. If G is a tree, then $\xi(G) \leq 2$, and the equality holds only when G is not a path [5]. Both $M(G) - Z_{\text{vc}}(G)$ and $\eta(G) - 1$ are lower bounds of $\xi(G)$ by Theorem 3.3.2 and [4]. When one of the lower bounds meets with the upper bound $\lfloor Z \rfloor(G)$, $\xi(G) = \lfloor Z \rfloor(G)$. Finally, if G has a T_3-family minor, then $\xi(G) \geq 3$ [14]. In this case, $\xi(G) = 3$ when $\lfloor Z \rfloor(G) = 3$. □

While $\xi(T) \leq 2$ for all tree T, the value of $\lfloor Z \rfloor(T)$ can be more than two. Example A.11. of [4] gives a tree T with $\lfloor Z \rfloor(T) = 3$; the graph T is shown in Figure 3.3. However, $\xi(G) = \lfloor Z \rfloor(G)$ is still true when G is a tree and $|V(G)| \leq 7$.

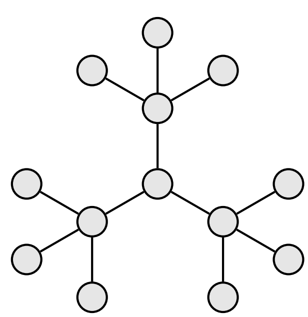


Figure 3.3 An example of tree T with $\lfloor Z \rfloor(T) = 3$

Lemma 3.4.2. *Let G be a tree with at most 7 vertices. Then $\xi(G) = \lfloor Z \rfloor(G)$.*

Proof. When G is a tree, it is known [5] that $\xi(G) = 2$ when G is not a path, and $\xi(G) = 1$ if G is a path. When G is a path, then $\xi(G) = 1 = \lfloor Z \rfloor(G)$. Assume G is not a path. It is enough to show $\lfloor Z \rfloor(G) \leq 2$. In this case, G must have a vertex v of degree at least 3. Call this type of vertex a high-degree vertex. If G has only one high degree vertex, then $\lfloor Z \rfloor(G) \leq 2$ since any two leaves form a ZFS-$\lfloor Z \rfloor$. Since $|V(G)| \leq 7$, there are at most two high-degree vertices. Pick two leaves such that the unique path between them contains only one high-degree vertex, then these two leaves form a ZFS-$\lfloor Z \rfloor$. □

Theorem 3.4.3. *Let G be a graph with at most 7 vertices. Then $\xi(G) = \lfloor Z \rfloor(G)$.*

Proof. Let G be a graph with at most 7 vertices. Then $M(G) = Z(G)$ [9]. If $Z_{\mathrm{SAP}}(G) = 0$, then $\xi(G) = M(G) = Z(G)$. Since $\xi(G) \leq \lfloor Z \rfloor(G) \leq Z(G)$, $\xi(G) = \lfloor Z \rfloor(G)$. If G is a tree, then $\xi(G) = \lfloor Z \rfloor(G)$ by Lemma 3.4.2. Then by Lemma 3.4.1, $\xi(G) = \lfloor Z \rfloor(G)$ for all connected graph G up to 7 vertices. It is known that $\xi(G_1 \dot\cup G_2) = \max\{\xi(G_1), \xi(G_2)\}$ [5] and $\lfloor Z \rfloor(G_1 \dot\cup G_2) = \max\{\lfloor Z \rfloor(G_1), \lfloor Z \rfloor(G_2)\}$ [4], so $\xi(G) = \lfloor Z \rfloor(G)$ for any graph up to 7 vertices. □

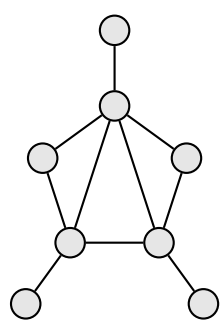


Figure 3.4 A graph G on 8 vertices with $\xi(G) = 2$ but $\lfloor Z \rfloor(G) = 3$

Example 3.4.4. Let G be the graph shown in Figure 3.4. It is known [17] that $M(G) = 2$. Since G is not a disjoint union of paths, $\xi(G) = 2$. Also, it can be computed that $Z(G) = \lfloor Z \rfloor(G) = 3$.

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CHAPTER 4. CONCLUSION

Zero forcing appears in many aspects of the minimum rank problem; even though the associated color-change rule varies in different cases, the core idea is the same—to control the nullity of a linear system.

In Chapter 2, the fact that $M(\mathfrak{C}^0_{2k+1}) = 0$ for loopless odd cycles $\mathfrak{C}^0_{2k+1}$ was used to introduce the odd cycle zero forcing number $Z_{oc}(\mathfrak{G})$ for loop graphs $\mathfrak{G}$. This new zero forcing number inserted a new parameter between the maximum nullity and the conventional zero forcing number; that is, $M(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$.

While $Z_{oc}(\mathfrak{G})$ and $\widehat{Z}_{oc}(G)$ provide upper bounds for the maximum nullity, there are many open questions regarding the lower bounds of the maximum nullity. The graph complement conjecture mentioned in Section 2.5 says

$$M(G) + M(\overline{G}) \geq n - 2$$

for all simple graphs on n vertices and is still open. On the other hand, Davila and Kenter [5] conjectured that

$$(g-3)(\delta-2) + \delta \leq Z(G)$$

for all simple graph G with girth $g \geq 3$ and minimum degree $\delta \geq 2$. Here the *girth* is the length of the shortest cycle on G; when G is a forest, the girth is set to be ∞, but the assumption $\delta \geq 2$ prevents the graph from being a forest. In 2016, Davila, Kalinowski, and Stephen [4] posted a proof of the conjecture. One may ask the same question for the maximum nullity; is it true that

$$(g-3)(\delta-2) + \delta \leq M(G)?$$

In the case of $g = 3$ or $\delta = 2$, this is known as the delta conjecture [2, 10], on which Hall provided a promising approach [7].

In Chapter 3, the primary focus was on the SAP. We defined the SAP matrix Ψ for a symmetric matrix A and observed that A has the SAP if and only if Ψ is full-rank. As a result, zero forcing was applied on the zero-nonzero pattern of Ψ to guarantee the SAP. Because the columns of the SAP matrix Ψ are indexed by the non-edges, the SAP zero forcing process is coloring the non-edges. With the help of $Z_{\text{SAP}}(G)$ and other theorems, the values of $\xi(G)$ for small graphs were found.

The zero forcing technique can be applied to all linear systems to control their nullities. There are some matrix properties similar to the SAP. A symmetric matrix A is said to have the Strong Spectral Property (SSP) if $X = O$ is the only real symmetric matrix that satisfies $A \circ X = I \circ X = AX - XA = O$; a symmetric matrix A is said to have the Strong Multiplicity Property (SMP) if $X = O$ is the only real symmetric matrix that satisfies $A \circ X = I \circ X = AX - XA = O$ and $\operatorname{tr}(A^i X) = 0$ for $i = 0, 1, \ldots, n-1$. The SMP and the SSP have many applications on the inverse eigenvalue problem of a graph [1]. One may apply similar techniques to those in Chapter 3 to design new parameters such that their vanishments guarantee the SSP or the SMP.

There are still many open questions regarding the minimum rank problem and the inverse eigenvalue problem of a graph. Zero forcing appears as an effective tool for these problems for small graphs or structured graphs, although for large random graphs the gap between the zero forcing number and the maximum nullity is inevitable [8, 9]. There should be more applications of zero forcing to be found.

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