

Zero forcing number, Grundy domination number, and their variants

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June 6, 2017

Abstract

This paper presents strong connections between four variants of the zero forcing number and four variants of the Grundy domination number. These connections bridge the domination problem and the minimum rank problem. We show that the Grundy domination type parameters are bounded above by the minimum rank type parameters. We also give a method to calculate the L -Grundy domination number by the Grundy total domination number, giving some linear algebra bounds for the L -Grundy domination number.

Keywords: Zero forcing, Grundy domination, minimum rank, maximum nullity

AMS subject classifications: 05C50, 05C57, 05C69, 05C70, 05C85.

1 Introduction

The zero forcing number $Z(G)$ considers a propagation process on a simple graph and refers to the minimum number of blue vertices so that all vertices can turn blue eventually under some color-change rule. The zero forcing number was introduced independently by mathematicians for studying the minimum rank problem [1] and physicists for quantum control [8]. The zero forcing number has been studied extensively, and many variants have been introduced; see, e.g., [3, 10] and the references therein. Variants of zero forcing have applications to the fast-mixed search in computer science [11] and the cops-and-robber game in graph theory [3]. Zero forcing can also be used for designing logic circuits [9].

On the other hand, a *dominating set* X of a simple graph G is a set of vertices such that every vertex outside of X has a neighbor in X , and the *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. One way to find

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a minimum dominating set is by the greedy algorithm: Start with an empty set X . Find a vertex v such that

$$N[v] \setminus \bigcup_{x \in X} N[x] \neq \emptyset$$

and add v to X , where $N[x]$ is the set of closed neighbors of x . Keep doing this step until no more vertex can be added to X . When the algorithm ends, it means $\bigcup_{x \in X} N[x] = V(G)$ and X is a dominating set. At any stage of the greedy algorithm, the elements in X can be written as a sequence (v_1, \dots, v_k) such that

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$$

for all $i = 1, \dots, k$. This condition guarantees that the newly added vertex is not redundant. Depending on the searching process of adding a new vertex, the greedy algorithm can either return a minimum dominating set or a larger set with the abovementioned structure. The Grundy domination number $\gamma_{\text{gr}}(G)$ refers to the maximum length of a sequence given by the greedy algorithm.

The Grundy domination number $\gamma_{\text{gr}}(G)$ and the Grundy total domination number $\gamma_{\text{gr}}^t(G)$ was introduced in [5] and [6], respectively. Brešar et al. [4] introduced the other two variants of the Grundy domination number, namely, the Z -Grundy domination number $\gamma_{\text{gr}}^Z(G)$ and the L -Grundy domination number; the authors also showed that $Z(G) + \gamma_{\text{gr}}^Z(G) = |V(G)|$, so finding the value of $\gamma_{\text{gr}}^Z(G)$ is equivalent to finding the value of $Z(G)$.

In this paper, we will show that not only the Z -Grundy domination number but every Grundy domination type parameters mentioned in [4] relates to a zero forcing type parameter. That is, we will show in Theorem 2.2 that

$$\begin{aligned} (1) \quad Z(G) &= n - \gamma_{\text{gr}}^Z(G), & (3) \quad Z_-(G) &= n - \gamma_{\text{gr}}^t(G), \\ (2) \quad Z_i(G) &= n - \gamma_{\text{gr}}(G), & (4) \quad Z_L(G) &= n - \gamma_{\text{gr}}^L(G), \end{aligned}$$

for any graph G . The definitions of these parameters will be introduced in Section 1.1.

In Section 2, we will prove the relations between the zero forcing type parameters and the Grundy domination type parameters, and many inequalities are given. Section 3 includes the definitions of the minimum rank type parameters, and we will show that each Grundy domination type parameters is bounded above by a minimum rank type parameter. Finally, in Section 4 we will provide a way to calculate the L -Grundy domination number by the Grundy total domination number and give some linear algebra bounds for $\gamma_{\text{gr}}^L(G)$. Figure 1 illustrates all related parameters, where a line connecting two parameters means the lower parameter is bounded above by the upper parameter.

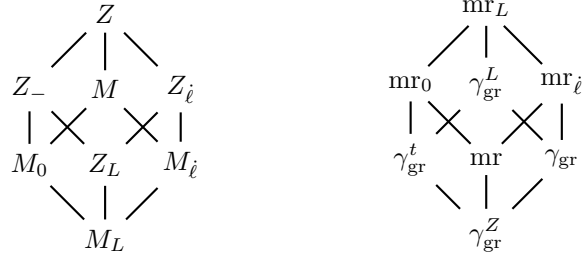


Figure 1: An illustration of related parameters

1.1 Preliminaries

All graph considered are simple and finite. For a vertex x of G , the *open neighborhood* of x in G is denoted as $N_G(x)$, while the *closed neighborhood* is denoted as $N_G[x]$. When the context is clear, we will simply use $N(x)$ and $N[x]$.

Let G be a graph. The *zero forcing game* is a color-change game such that each vertex is colored blue or white initially, and then the *color change rule* (CCR) is applied repeatedly. The color change rule varies from different variants of the zero forcing game, as we will see in Definition 1.1. If starting with an initial blue set $B \subseteq V(G)$ makes every vertex turn blue eventually, then B is called a *zero forcing set*. The *zero forcing number* is defined as the minimum cardinality of a zero forcing set.

Different types of zero forcing numbers have been discussed in the literature (e.g., see [3, 13]). As we will see in Section 3, many of them serve as upper bounds for variants of the maximum nullity. Here we recall three types of the zero forcing numbers $Z(G)$, $Z_i(G)$, $Z_-(G)$ from [1, 13, 14] and introduce a new parameter, the *L-zero forcing number* $Z_L(G)$.

Definition 1.1. On a graph where vertices are colored blue or white, the color change rule for each of Z , Z_i , Z_- , and Z_L are as follows.

- (1) (CCR- Z) If $y \in N(x)$ and $N[x]$ are all blue except for y , then y turns blue. Denoted as $x \xrightarrow{Z} y$.
- (2) (CCR- Z_i) If $y \in N[x]$ and $N[x]$ are all blue except for y , then y turns blue. Denoted as $x \xrightarrow{Z_i} y$.
- (3) (CCR- Z_-) If $y \in N(x)$ and $N(x)$ are all blue except for y , then y turns blue. Denoted as $x \xrightarrow{Z_-} y$.
- (4) (CCR- Z_L) Either $x \xrightarrow{Z_-} y$ when $x \neq y$, or $x \xrightarrow{Z_i} y$ when $x = y$ may apply. Denoted as $x \xrightarrow{Z_L} y$.

One may think of CCR- Z_L as the following: If $y \in N[x]$ and $N(x)$ are all blue “except” for y , then y turns blue. However, this definition is not clear in the case of $x = y$, so we explicitly separate it into two cases: $x \xrightarrow{Z_-} y$ when $x \neq y$, or $x \xrightarrow{Z_\ell} y$ when $x = y$.

Remark 1.2. In [13], the zero forcing number of a loop graph is defined. The parameter $Z_\ell(G)$ is the same as the zero forcing number of a loop graph that is obtained from G by considering every vertex as having a loop. Notice that $Z_\ell(G)$ is slightly different from the loop zero forcing number $Z_\ell(G)$ defined in [3]. When G has exactly r isolated vertices, $Z_\ell(G) = Z_\ell(G) + r$.

Similarly, the parameter $Z_-(G)$ is the same as the zero forcing number of a loop graph that is obtained from G by considering every vertex as having no loop. It is also called the skew zero forcing number in [14] for studying the minimum rank problem on skew-symmetric matrices.

When the color change rule in effect is clear, we sometimes omit the superscript above the arrow and write $a \rightarrow b$ as a force. In a zero forcing game with a given color change rule, the *chronological list* records the performed forces $a_i \rightarrow b_i$ in the chronological order. A game is called *successful* if all vertices turn blue at the end. If $(a_i \rightarrow b_i)_{i=1}^k$ is the chronological list of a successful zero forcing game, then b_1, \dots, b_k are the initial white vertices and $V(G) \setminus \{b_1, \dots, b_k\}$ is the set of initial blue vertices, which is a zero forcing set.

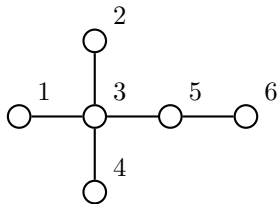


Figure 2: An example where Z , Z_- , Z_ℓ , and Z_L are all distinct

Example 1.3. Let G be the graph as in Figure 2. Then $Z(G) = 3$, $Z_\ell(G) = 1$, $Z_-(G) = 2$, and $Z_L(G) = 0$. For each zero forcing type parameters, a minimum zero forcing set B along with its chronological list is shown below.

$Z(G) = 3$ $B = \{1, 2, 4\}$	$Z_\ell(G) = 1$ $B = \{3\}$	$Z_-(G) = 2$ $B = \{1, 2\}$	$Z_L(G) = 0$ $B = \{\}$
$1 \xrightarrow{Z} 3$	$1 \xrightarrow{Z_\ell} 1$	$4 \xrightarrow{Z_-} 3$	$1 \xrightarrow{Z_L} 3$
$3 \xrightarrow{Z} 5$	$2 \xrightarrow{Z_\ell} 2$	$5 \xrightarrow{Z_-} 6$	$1 \xrightarrow{Z_L} 1$
$5 \xrightarrow{Z} 6$	$4 \xrightarrow{Z_\ell} 4$	$6 \xrightarrow{Z_-} 5$	$2 \xrightarrow{Z_L} 2$
	$3 \xrightarrow{Z_\ell} 5$	$3 \xrightarrow{Z_-} 4$	$4 \xrightarrow{Z_L} 4$
	$5 \xrightarrow{Z_\ell} 6$		$3 \xrightarrow{Z_L} 5$
			$5 \xrightarrow{Z_L} 6$

Definition 1.4. Let G be a graph. A sequence (v_1, v_2, \dots, v_k) of distinct vertices of G is called

- (1) a *Z-sequence* if $N(v_i) \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$ for all $i = 1, \dots, k$;
- (2) a *dominating sequence* if $N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$ for all $i = 1, \dots, k$;
- (3) a *total dominating sequence* if $N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ for all $i = 1, \dots, k$;
- (4) an *L-sequence* if $N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ for all $i = 1, \dots, k$.

The *Z-Grundy domination number* $\gamma_{\text{gr}}^Z(G)$ is the largest length of a *Z-sequence*; the *Grundy domination number* $\gamma_{\text{gr}}(G)$ is the largest length of a dominating sequence; the *Grundy total domination number* $\gamma_{\text{gr}}^t(G)$ is the largest length of a total dominating sequence; the *L-Grundy domination number* $\gamma_{\text{gr}}^L(G)$ is the largest length of a *L-sequence*.

Example 1.5. Let G be the graph in Figure 2. Then $(6, 5, 3)$ is a *Z-sequence*. Indeed, for each chronological list shown in Example 1.3, reading the second column from the bottom to the top gives the corresponding sequence. That is, $(6, 5, 4, 2, 1)$ is a dominating sequence, $(4, 5, 6, 3)$ is a total dominating sequence, and $(6, 5, 4, 2, 1, 3)$ is an *L-sequence*.

Remark 1.6. In [4], the *Z-Grundy domination number* and the *Grundy total domination number* are defined on graphs without isolated vertices. However, the same definitions are well-defined for all graph. Indeed, if a graph G can be written as $H \dot{\cup} rK_1$ such that H has no isolated vertices, then $\gamma_{\text{gr}}^Z(G) = \gamma_{\text{gr}}^Z(H)$ and $\gamma_{\text{gr}}^t(G) = \gamma_{\text{gr}}^t(H)$.

2 The four zero forcing type parameters and the four Grundy domination type parameters

In this section we give the relations between the four zero forcing type parameters Z , Z_ℓ , Z_- , Z_L and the four Grundy domination type parameters γ_{gr}^Z , γ_{gr} , γ_{gr}^t , γ_{gr}^L . Note that the identity $Z(G) + \gamma_{\text{gr}}^Z(G) = |V(G)|$ has been shown in [4].

Lemma 2.1. *Let G be a graph and (b_1, b_2, \dots, b_k) a sequence of vertices of G . There exist vertices a_1, a_2, \dots, a_k such that $(a_i \rightarrow b_i)_{i=1}^k$ is the chronological list of a successful zero forcing game with*

- | | |
|--------------------------|-----------------------------|
| (1) CCR-Z, | (3) CCR-Z ₋ , or |
| (2) CCR-Z _i , | (4) CCR-Z _L |

if and only if $(b_k, b_{k-1}, \dots, b_1)$ is

- | | |
|----------------------------|-------------------------------------|
| (1) a Z-sequence, | (3) a total dominating sequence, or |
| (2) a dominating sequence, | (4) a L-sequence, |

respectively.

Proof. Case (1) is in [4]. Here we prove Case (4), as the others are similar.

Suppose there are vertices a_1, a_2, \dots, a_k such that $(a_i \xrightarrow{Z_L} b_i)_{i=1}^k$ is the chronological list of a successful zero forcing game with CCR-Z_L. We will claim that

$$a_i \in N[b_i] \setminus \bigcup_{j=i+1}^k N(b_j)$$

for all i . Consequently, $(b_k, b_{k-1}, \dots, b_1)$ is an L-sequence. For each $i = 1, \dots, k$, it must be one of the two cases following.

Case A: $a_i = b_i$ and $a_i \xrightarrow{Z_L} b_i$ is done through $a_i \xrightarrow{Z_i} b_i$. This means by the time $a_i \xrightarrow{Z_L} b_i$ all vertices in $N(a_i)$ are blue and b_i is white. At this moment, the white vertices are b_i, b_{i+1}, \dots, b_k , so $b_j \notin N(a_i)$ for all $j = i+1, \dots, k$, and $a_i \notin \bigcup_{j=i+1}^k N(b_j)$. Also, $a_i = b_i \in N[b_i]$.

Case B: $b_i \in N(a_i)$ and $a_i \xrightarrow{Z_L} b_i$ is done through $a_i \xrightarrow{Z_-} b_i$. This means by the time $a_i \xrightarrow{Z_L} b_i$ all vertices in $N(a_i)$ are blue except for b_i . At this moment, the white vertices are b_i, b_{i+1}, \dots, b_k , so $b_j \notin N(a_i)$ for all $j = i+1, \dots, k$, and $a_i \notin \bigcup_{j=i+1}^k N(b_j)$.

Conversely, suppose $(b_k, b_{k-1}, \dots, b_1)$ is an L-sequence. Pick a_i as an element in $N[b_i] \setminus \bigcup_{j=i+1}^k N(b_j)$ for each i . We will show that $(a_i \xrightarrow{Z_L} b_i)_{i=1}^k$ is the chronological list of a successful zero forcing game with CCR-Z_L, starting with the initial blue set

$$V(G) \setminus \{b_1, b_2, \dots, b_k\}.$$

To see this, assume at the i -th step the vertices b_1, \dots, b_{i-1} are blue, or equivalently, the white vertices are b_i, \dots, b_k .

Case A: $a_i = b_i$ and $a_i \notin \bigcup_{j=i+1}^k N(b_j)$. This means $b_j \notin N(a_i)$ for all $j = i+1, \dots, k$, and $N(a_i)$ are all blue. Therefore, $a_i \xrightarrow{Z_i} b_i$ applies.

Case B: $a_i \in N(b_i)$ and $a_i \notin \bigcup_{j=i+1}^k N(b_j)$. This means $b_j \notin N(a_i)$ for all $j = i+1, \dots, k$, and $N(a_i)$ are all blue except for b_i . Thus, b_i is the only white vertex in $N(a_i)$, and $a_i \xrightarrow{Z_-} b_i$ applies. \square

Theorem 2.2. *Let G be a graph and $|V(G)| = n$. Then*

$$\begin{aligned} (1) \quad Z(G) &= n - \gamma_{\text{gr}}^Z(G), & (3) \quad Z_-(G) &= n - \gamma_{\text{gr}}^t(G), \\ (2) \quad Z_{\hat{i}}(G) &= n - \gamma_{\text{gr}}(G), & (4) \quad Z_L(G) &= n - \gamma_{\text{gr}}^L(G). \end{aligned}$$

Proof. This follows immediately from Lemma 2.1. □

Proposition 2.3. *Let G be a graph on n vertices. Then*

$$\begin{aligned} (1) \quad Z_L(G) &\leq Z_{\hat{i}}(G) \leq Z(G), & (3) \quad 2Z(G) &\leq n + Z_-(G), \text{ and} \\ (2) \quad Z_L(G) &\leq Z_-(G) \leq Z(G), & (4) \quad 2Z_{\hat{i}}(G) &\leq n + Z_L(G). \end{aligned}$$

Moreover, $Z_L(G) + 1 \leq Z_{\hat{i}}(G)$ if G has at least an edge.

Proof. For (1) and (2), the inequalities follow from the definition.

Suppose G has exactly r isolated vertices and is written as $H \dot{\cup} rK_1$, where H does not have any vertex if G has no edge.

If G has at least an edge, then H is a (non-degenerated) graph without isolated vertices. By [4], $\gamma_{\text{gr}}(H) \leq \gamma_{\text{gr}}^L(H) - 1$, so

$$\begin{aligned} Z_L(G) + 1 &= n - \gamma_{\text{gr}}^L(G) + 1 = n - \gamma_{\text{gr}}^L(H) - r + 1 \\ &\leq n - \gamma_{\text{gr}}(H) - r = n - \gamma_{\text{gr}}(G) = Z_{\hat{i}}(G) \end{aligned}$$

by Theorem 2.2.

For (3), it is known [4] that $\gamma_{\text{gr}}^t(H) \leq 2\gamma_{\text{gr}}^Z(H)$ for any graph without isolated vertices, so

$$\begin{aligned} 2Z(G) &= 2n - 2\gamma_{\text{gr}}^Z(G) = 2n - 2\gamma_{\text{gr}}^Z(H) \\ &\leq 2n - \gamma_{\text{gr}}^t(H) = 2n - \gamma_{\text{gr}}^t(G) = n + Z_-(G) \end{aligned}$$

by Theorem 2.2.

For (4), it is known [4] that $\gamma_{\text{gr}}^L(G) \leq 2\gamma_{\text{gr}}(G)$ for any graph, so

$$2Z_{\hat{i}}(G) = 2n - 2\gamma_{\text{gr}}(G) \leq 2n - \gamma_{\text{gr}}^L(G) = n + Z_L(G)$$

by Theorem 2.2. □

Recall that $\gamma(G)$ is the domination number. Since $\gamma_{\text{gr}}(G)$ considers the worst case in the greedy algorithm of finding $\gamma(G)$, we know $\gamma(G) \leq \gamma_{\text{gr}}(G)$. Indeed, for a graph G without isolated vertices, any maximal Z -sequence also gives a dominating set, so $\gamma(G) \leq \gamma_{\text{gr}}^Z(G)$. This gives an upper bound to $Z(G)$.

Proposition 2.4. *Let G be a graph without isolated vertex and $n = |V(G)|$. Then $Z(G) \leq n - \gamma(G)$.*

3 The minimum rank type parameters

Let G be a graph on n vertices. Define $\mathcal{S}(G)$ as the family of $n \times n$ real symmetric matrices whose i, j -entry, $i \neq j$, is nonzero whenever $\{i, j\} \in E(G)$ and zero otherwise. Notice that there are no restrictions on the diagonal entries. The *maximum nullity* and the *minimum rank* of G are defined as

$$M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\} \text{ and}$$

$$\text{mr}(G) = \min\{\text{rank}(A) : A \in \mathcal{S}(G)\},$$

respectively. By definition, $M(G) + \text{mr}(G) = |V(G)|$. It is shown in [1] that $M(G) \leq Z(G)$ for every graph.

Among matrices in $\mathcal{S}(G)$, let $\mathcal{S}_\ell(G)$ be the matrices with every diagonal entry nonzero, and let $\mathcal{S}_0(G)$ be the matrices with every diagonal entry zero. Similarly, define

$$M_\ell(G) = \max\{\text{null}(A) : A \in \mathcal{S}_\ell(G)\},$$

$$\text{mr}_\ell(G) = \min\{\text{rank}(A) : A \in \mathcal{S}_\ell(G)\},$$

$$M_0(G) = \max\{\text{null}(A) : A \in \mathcal{S}_0(G)\},$$

$$\text{mr}_0(G) = \min\{\text{rank}(A) : A \in \mathcal{S}_0(G)\}.$$

By definition, $M_\ell(G) + \text{mr}_\ell(G) = |V(G)|$ and $M_0(G) + \text{mr}_0(G) = |V(G)|$. Also, it is known [13] that $M_\ell(G) \leq Z_\ell(G)$ and $M_0(G) \leq Z_-(G)$.

Usually, finding a lower bound for a Grundy domination type parameter is by constructing a sequence and verify if the sequence has the desired property. On the other side, finding an upper bound for a Grundy domination type parameter requires an argument showing every sequence with the corresponding properties cannot be too long. Theorem 3.1 gives a fairly easy way to find upper bounds for the Grundy domination type parameters. (The upper bound for the L -Grundy domination number will be provided in Section 4.)

Theorem 3.1. *Let G be a graph. Then*

- (1) $\gamma_{\text{gr}}^Z(G) \leq \text{mr}(G)$,
- (2) $\gamma_{\text{gr}}(G) \leq \text{mr}_\ell(G)$, and
- (3) $\gamma_{\text{gr}}^t(G) \leq \text{mr}_0(G)$.

Proof. By [1, 13], $M(G) \leq Z(G)$, $M_\ell(G) \leq Z_\ell(G)$, and $M_0(G) \leq Z_-(G)$. This is equivalent to $\gamma_{\text{gr}}^Z(G) \leq \text{mr}(G)$, $\gamma_{\text{gr}}(G) \leq \text{mr}_\ell(G)$, and $\gamma_{\text{gr}}^t(G) \leq \text{mr}_0(G)$ by Theorem 2.2. \square

Example 3.2. Let P be the Petersen graph as drawn in Figure 3. Then $(1, 2, 3, 4, 5)$ is a Z -sequence and also a dominating sequence. And $(9, 1, 2, 3, 4, 5)$ is a total dominating sequence.

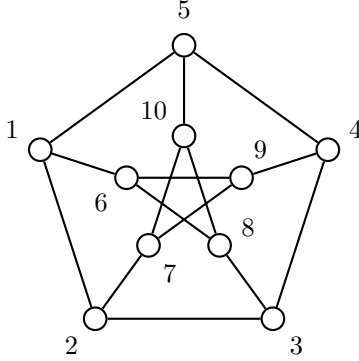


Figure 3: A labeled Petersen graph

Let

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } C' = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} C - I_5 & I_5 \\ I_5 & C' - I_5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -C & I_5 \\ I_5 & C' \end{bmatrix},$$

where I_5 is the identity matrix of order 5. One may check that $A \in \mathcal{S}_\ell(G) \subseteq \mathcal{S}(G)$ and $\text{rank}(A) = 5$; also, $B \in \mathcal{S}_0(G)$ and $\text{rank}(B) = 6$. Therefore, we know $\gamma_{\text{gr}}^Z(G) = \gamma_{\text{gr}}(G) = 5$ and $\gamma_{\text{gr}}^t(G) = 6$.

Recall that the *independence number* $\alpha(G)$ is the largest cardinality of a independent set, and the *vertex cover number* $\beta(G)$ is the minimum number of vertices such that every edge is incident to at least one of these vertices. It is a standard result that $\alpha(G) + \beta(G) = |V(G)|$. In [6], it was shown that $\gamma_{\text{gr}}^t(G) \leq 2\beta(G)$. Here we improve this result by showing $\text{mr}_0(G) \leq 2\beta(G)$.

Proposition 3.3. *Let G be a graph. Then $Z_-(G) \geq M_0(G) \geq \alpha(G) - \beta(G)$ and $\gamma_{\text{gr}}^t(G) \leq \text{mr}_0(G) \leq 2\beta(G)$.*

Proof. Let A be a matrix in $\mathcal{S}_0(G)$ and X an independent set of G with $|X| = \alpha(G)$. Since there is no edges in X , the submatrix $A[X]$ of A induced on rows and columns in X is a zero matrix, and $\text{rank}(A[X]) = 0$. Also, the matrix A can be obtained from $A[X]$ by adding $\beta(G)$ rows and $\beta(G)$ columns, since $|V(G)| - \alpha(G) = \beta(G)$. Adding a row or a column can increase the rank by at most one, so

$$\text{rank}(A) \leq \text{rank}(A[X]) + 2\beta(G) = 2\beta(G)$$

and $\text{mr}_0(G) \leq \text{rank}(A) \leq 2\beta(G)$. Other inequalities follows from Theorem 2.2 and Theorem 3.1. \square

The *edge clique cover number* $cc(G)$ is the minimum number of cliques that can cover every edge of G . It is noted in [4] that $\gamma_{gr}(G) \leq cc(G)$. Here we prove that $mr_{\hat{\ell}}(G) \leq cc(G)$; the technique is standard in the field of the minimum rank problem.

Proposition 3.4. *Let G be a graph. Then $Z_{\hat{\ell}}(G) \geq M_{\hat{\ell}}(G) \geq n - cc(G)$ and $\gamma_{gr}(G) \leq mr_{\hat{\ell}}(G) \leq cc(G)$.*

Proof. Let $k = cc(G)$ and $\{C_1, \dots, C_k\}$ an edge clique cover. For each clique C_t we may construct a rank-one matrix A_t where the i, j -entry is 1 whenever both i and j ($i = j$ is possible) are in C_t and zero otherwise. Thus, the matrix $A = \sum_{t=1}^k A_t$ is a matrix in $\mathcal{S}_{\hat{\ell}}(G)$ with $\text{rank}(A) \leq k$. Therefore, $mr_{\hat{\ell}}(G) \leq k = cc(G)$. Other inequalities follows from Theorem 2.2 and Theorem 3.1. \square

4 Linear algebra bounds for the L -Grundy domination number

In this section, we will provide a relation between the L -Grundy domination number and the Grundy total domination number. Then we use this relation to provide some linear algebra bounds for the L -Grundy domination number.

Let G be a graph on the vertex set $\{1, \dots, n\}$. Construct a bipartite graph $\mathcal{B}_L(G)$ with

- $V(\mathcal{B}_L(G)) = \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\} \cup \{z_1, \dots, z_n\}$,
- $E(\mathcal{B}_L(G)) = \{\{x_i, y_j\}, \{x_i, z_j\} : \{i, j\} \in E(G)\} \cup \{\{x_i, y_i\} : i \in V(G)\}$.

Let G be a graph and $X \subseteq V(G)$. We say $\gamma_{gr}^t(G, X)$ is the maximum length of a total dominating sequence of G using only vertices in X .

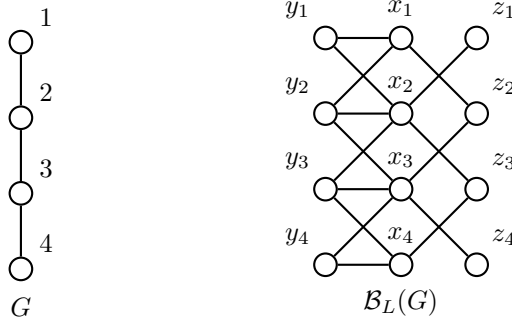


Figure 4: An illustration of G and $\mathcal{B}_L(G)$

Example 4.1. Let G be the path on 4 vertices. Then the graph $\mathcal{B}_L(G)$ is as shown in Figure 4. The graph G has $\gamma_{\text{gr}}^L(G) = 4$ and $Z_L(G) = 0$, as $(4, 3, 1, 2)$ is a maximum L -sequence in G .

At the same time, (x_4, x_3, x_1, x_2) is a total dominating sequence in $\mathcal{B}_L(G)$, so $\gamma_{\text{gr}}^t(G, X) = 4$ for $X = \{x_1, x_2, x_3, x_4\}$.

When G is a bipartite graph, its total dominating sequence has a nice decomposition, as shown in Proposition 8.3 and Theorem 8.4 of [6].

Proposition 4.2. *Let G be a bipartite graph with two parts X and Y . Then $\gamma_{\text{gr}}^t(G) = 2\gamma_{\text{gr}}^t(G, X) = 2\gamma_{\text{gr}}^t(G, Y)$.*

Proof. Any bipartite graph G can be the incidence graph of a hypergraph \mathcal{H} ; that is, X represents the vertices of \mathcal{H} , Y represents the edges of \mathcal{H} , and there is an edge between $x \in X$ and $y \in Y$ if and only if vertex x is incident to edge y in \mathcal{H} . Following the notation in [6], we have $\gamma_{\text{gr}}^t(G, X) = \tau_{\text{gd}}(\mathcal{H})$ and $\gamma_{\text{gr}}^t(G, Y) = \rho_{\text{gd}}(\mathcal{H})$, so the desired results follow from Proposition 8.3 and Theorem 8.4 of [6]. \square

Theorem 4.3. *Let G be a graph on n vertices. Then $Z_-(\mathcal{B}_L(G)) = n + 2Z_L(G)$ and $\gamma_{\text{gr}}^t(\mathcal{B}_L(G)) = 2\gamma_{\text{gr}}^L(G)$.*

Proof. Let $X = \{x_1, \dots, x_n\}$. We will show that $\gamma_{\text{gr}}^t(\mathcal{B}_L(G), X) = \gamma_{\text{gr}}^L(G)$. Then the desired results follow from Proposition 4.2 and Theorem 2.2.

Let $k = \gamma_{\text{gr}}^L(G)$ and (v_1, \dots, v_k) an L -sequence of G . We can verify that $(x_{v_1}, \dots, x_{v_k})$ is a total dominating sequence of $\mathcal{B}_L(G)$. Since (v_1, \dots, v_k) is an L -sequence in G , we may pick

$$u_i \in N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G(v_j)$$

for each i with $1 \leq i \leq k$. If $u_i \neq v_i$, then

$$z_{u_i} \in N_{\mathcal{B}_L(G)}(x_{v_i}) \setminus \bigcup_{i=1}^{i-1} N_{\mathcal{B}_L(G)}(x_{v_j}).$$

If $u_i = v_i$, then

$$y_{u_i} \in N_{\mathcal{B}_L(G)}(x_{v_i}) \setminus \bigcup_{i=1}^{i-1} N_{\mathcal{B}_L(G)}(x_{v_j}).$$

Therefore, $(x_{v_1}, \dots, x_{v_k})$ is a total dominating sequence of $\mathcal{B}_L(G)$, and

$$\gamma_{\text{gr}}^t(\mathcal{B}_L(G), X) \geq k = \gamma_{\text{gr}}^L(G).$$

Conversely, suppose $h = \gamma_{\text{gr}}^t(\mathcal{B}_L(G), X)$ and $(x_{v_1}, \dots, x_{v_h})$ is a total dominating sequence of $\mathcal{B}_L(G)$. We can verify that (v_1, \dots, v_h) is an L -sequence of

G . Since $(x_{v_1}, \dots, x_{v_h})$ is a total dominating sequence of $\mathcal{B}_L(G)$, we may pick an element in

$$N_{\mathcal{B}_L(G)}(x_{v_i}) \setminus \bigcup_{j=1}^{i-1} N_{\mathcal{B}_L(G)}(x_{v_j})$$

and let u_i be its index. Thus

$$u_i \in N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G(v_j).$$

Therefore, (v_1, \dots, v_h) is an L -sequence of G , and $\gamma_{\text{gr}}^L(G) \geq h = \gamma_{\text{gr}}^t(\mathcal{B}_L(G), X)$. This completes the proof. \square

Corollary 4.4. *Let G be a graph on n vertices. Then*

$$\frac{M_0(\mathcal{B}_L(G)) - n}{2} \leq Z_L(G) \text{ and } \gamma_{\text{gr}}^L(G) \leq \frac{1}{2} \text{mr}_0(\mathcal{B}_L(G)).$$

Proof. By Theorem 4.3 and Theorem 3.1,

$$\gamma_{\text{gr}}^L(G) = \frac{1}{2} \gamma_{\text{gr}}^t(\mathcal{B}_L(G)) \leq \frac{1}{2} \text{mr}_0(\mathcal{B}_L(G)).$$

For the other inequality follows,

$$\begin{aligned} Z_L(G) &= n - \gamma_{\text{gr}}^L(G) \geq n - \frac{1}{2} \text{mr}_0(\mathcal{B}_L(G)) \\ &= n - \frac{1}{2}(3n - M_0(\mathcal{B}_L(G))) = \frac{M_0(\mathcal{B}_L(G)) - n}{2} \end{aligned}$$

by Theorem 2.2. \square

For a graph G on n vertices, define $\text{mr}_L(G)$ as the minimum rank over matrices of the form $\begin{bmatrix} A & B \end{bmatrix}$ with $A \in \mathcal{S}_i(G)$ and $B \in \mathcal{S}_0(G)$, and let $M_L(G) = n - \text{mr}_L(G)$.

Corollary 4.5. *For any graph G , $M_L(G) \leq Z_L(G)$ and $\gamma_{\text{gr}}^L(G) \leq \text{mr}_L(G)$.*

Proof. Let $A \in \mathcal{S}_i(G)$ and $B \in \mathcal{S}_0(G)$ be matrices such that $\text{rank}(\begin{bmatrix} A & B \end{bmatrix}) = \text{mr}_L(G)$. Then the matrix

$$\begin{bmatrix} O & A & B \\ A & O & O \\ B & O & O \end{bmatrix}$$

is a matrix in $\mathcal{S}_0(\mathcal{B}_L(G))$ with rank $2 \text{mr}_L(G)$. Therefore, by Corollary 4.4

$$\gamma_{\text{gr}}^L(G) \leq \frac{1}{2} \text{mr}_0(\mathcal{B}_L(G)) \leq \text{mr}_L(G).$$

The other inequality follows from Theorem 2.2. \square

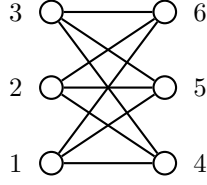


Figure 5: A labeled $K_{3,3}$

Example 4.6. Let G be the complete bipartite graph $K_{3,3}$ as shown in Figure 5. Then $(1, 2, 3, 4)$ is an L -sequence. The rectangular matrix

$$C = \begin{bmatrix} 3 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & -2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 3 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

is of the form $[A \ B]$ with $A \in \mathcal{S}_i(G)$ and $B \in \mathcal{S}_0(G)$. Since $\text{rank}(C) = 4$, the sequence $(1, 2, 4, 5)$ is a maximum L -sequence.

5 Conclusion

In this paper we found the relations between the four zero forcing type parameters Z , Z_i , Z_- , Z_L and the four Grundy domination type parameters γ_{gr}^Z , γ_{gr} , γ_{gr}^t , γ_{gr}^L . With these relation we are allowed to bring results from one sides to the other; for example, Theorem 3.1 provides upper bounds of the Grundy domination type parameters due to the application of the zero forcing type parameters to the minimum rank problem.

There are yet more zero forcing type parameters; see, e.g., [3, 15, 16]. In [2], the positive semi-definite zero forcing number $Z_+(G)$ was shown to have the relation $Z_+(G) + \text{OS}(G) = |V(G)|$. Here $\text{OS}(G)$ is the *OS-number* introduced in [12] with its motivation from linear algebra and defined as the largest length of a sequence (v_1, \dots, v_k) such that

$$N(v_i) \setminus \bigcup_{\substack{x \in V(H_i) \\ x \neq v_i}} N[x],$$

where H_i is the connected component of the induced subgraph $G[\{v_1, \dots, v_i\}]$ that contains v_i . With this definition, one may view the OS-number as an Grundy domination type parameters, and it would be interesting to see if there are further application of the OS-number to the domination problem.

On the other side, Brešar, Klavžar, and Rall [7] introduced the domination game as follows. Two players, Dominator and Staller, are picking vertices alternatively; each one needs to pick a vertex that dominates at least a new vertex that is not dominated by any previous chosen vertices. Dominator wants to dominate all vertices as fast as possible, while Staller is trying to slow down the process. The *game domination number* $\gamma_g(G)$ is the number of required steps to finish the game if starting with Dominator. The game domination number and its variants should also have their counterparts as zero forcing type parameters, and it would be nice to see their applications to the minimum rank problem.

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