# The sieving process and lower bounds for the minimum rank problem


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**Abstract**

For a given simple graph G, we associate a family $\mathcal{S}(G)$ of real symmetric matrices whose $i,j$-entry, $i \neq j$, is nonzero whenever $ij$ is an edge, and zero otherwise. We put no restriction on the diagonal entries. The *minimum rank problem* is to determine the *minimum rank* $\mathrm{mr}(G)$ among matrices in $\mathcal{S}(G)$, or equivalently, to determine the *maximum nullity* $M(G)$ among matrices $\mathcal{S}(G)$. The *zero forcing number* $Z(G)$ and the *enhanced zero forcing number* $\widehat{Z}(G)$ of a graph $G$ were introduced as upper bounds of $M(G)$; $M(G) \leq \widehat{Z}(G) \leq Z(G)$ for all graphs $G$ [1, 3]. However, $\widehat{Z}(G)$ is in general hard to compute, and examples show that $M(G) < \widehat{Z}(G)$ is possible. The purpose of this paper is to provide a method, the *sieving process*, to analyze the zero-nonzero patterns on the diagonal entries of optimal matrices, and use it to get new upper bounds for $M(G)$. This technique can also simplify the computation of $\widehat{Z}(G)$. Most graphs for which it is known that $M(G) < Z(G)$ can be explained by this technique; at the same time, the diagonal patterns become clear through this process.



**Keywords:** minimum rank, maximum nullity, zero forcing number, enhanced zero forcing number, sieving process, elimination lemma.


## 1 Introduction

For a given simple graph $G = (V(G), E(G))$, define a family $\mathcal{S}(G)$ of $n \times n$ real symmetric matrices whose $i,j$-entry, $i \neq j$, is:

- zero if $i$ is not adjacent to $j$;


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- nonzero if $i$ is adjacent to $j$.

The *minimum rank* $\operatorname{mr}(G)$ is defined as the smallest possible rank among matrices in $\mathcal{S}(G)$, and the *maximum nullity* $M(G)$ is defined as the largest possible rank among the same set of matrices. The *minimum rank problem* is to determine the minimum rank $\operatorname{mr}(G)$ for a given graph $G$, or equivalently, to determine the *maximum nullity* $M(G)$ by the fact that $\operatorname{mr}(G) + M(G) = |V(G)|$.

The main goal of this article is to use the sieving process to find bounds for the minimum rank problem, and to achieve this goal, we are forced to look at graphs containing loops or more than one edge between two vertices; this terminology will be discussed in detail later.

In 2008, AIM minimum rank-special graphs work group [1] introduced the *zero forcing number* $Z(G)$, defined as follows, as an upper bound of $M(G)$. The *zero forcing process* (for simple graphs) is a color-change rule such that:

- a set $B$ of vertices are set blue initially, while others remain white;
- if a vertex $x$ is blue and has exactly one white neighbor $y$, $y$ becomes blue at next step.

If the set $B$ can force all vertices to blue, it is called a *zero forcing set*. The zero forcing number $Z(G)$ is defined as the minimum cardinality of a zero forcing set, and was proved [1] to have the relation

$$M(G) \leq Z(G),$$

which is equivalent to

$$\operatorname{mr}(G) \geq |V(G)| - Z(G).$$

So finding a lower bound for $\operatorname{mr}(G)$ between $\operatorname{mr}(G)$ and $|V(G)| - Z(G)$ is the same with finding an upper bound for $M(G)$ between $M(G)$ and $Z(G)$, and we state the results for $M(G)$, instead of $\operatorname{mr}(G)$. Also, zero forcing as just defined was independently introduced by mathematical physicists [7] for control of quantum systems. Variants of zero forcing are cops and robbers games [3].

Even though $Z(G)$ agrees with $M(G)$ for small graphs [10] and for many structured graphs [1], there are still many graphs $G$ with $M(G) < Z(G)$. Some are listed in Table 1, and in fact these parameters diverge for large random graphs [12, 15]. To improve the bound, Barioli et. al. [3] proposed the *enhanced zero forcing number*, which will be defined later, by considering the zero forcing process on looped graphs. On the other hand, Hein van der Holst [13] provided a reduction formula for $M(G)$ by considering the maximum nullity of multigraphs. These results indicate a generalization to looped multigraphs is desirable.

We use the notation $e(i,j) = e(j,i)$ to denote the number of edges between vertex $i$ and $j$. Unless stated otherwise, $i$ and $j$ are not necessarily different vertices. In the case $i = j$, this edge with two endpoints on $i$ is called a *loop*, and we write $e(i) := e(i,i)$ as the number of loops on $i$.

To avoid ambiguity, all types of graphs to be considered are now defined. A *simple graph* is a graph where $0 \geq e(i,j) \leq 1$ for $i \neq j$ and $e(i) = 0$; a *multigraph* is a graph where $e(i,j)$ can be greater than 1 for $i \neq j$ and $e(i) = 0$. Also, a *looped graph* is a graph where $0 \leq e(i,j) \leq 1$ for $i \neq j$ yet $e(i)$ can be greater than 1; a *looped multigraph* is a graph where $e(i,j)$, including $e(i)$, can be greater than 1. Note that our looped graphs allow two or more loops on a vertex. The neighborhood of a vertex $i$ is $N(i) := \{j : e(i,j) > 0\}$. So a vertex with a loop on it is a neighbor of itself, and the number of neighbors does not count multiplicity. That is, no matter what is the value of $e(i,j)$, $j$ is *one* neighbor of $i$ whenever $e(i,j) > 0$. Since it looks the same between a simple graph and a looped graph without loops, throughout this article, a simple graph is denoted as $G$ in general, and a graph that allows loops is denoted as $\widehat{G}$. These agreements enable us to discuss our problems with ease.

Now we turn to the minimum rank and the zero forcing number of looped multigraphs. The definitions below are simply a combination of those in previous works [3, 13].

For a given looped multigraph $\widehat{G}$, define a family $\mathcal{S}(\widehat{G})$ of $n \times n$ real symmetric matrix whose $i,j$-entry, where $i,j$ are not necessarily distinct, is:

- zero if $e(i,j) = 0$ in $\widehat{G}$;
- nonzero if $e(i,j) = 1$ in $\widehat{G}$;
- any real number if $e(i,j) \geq 2$ in $\widehat{G}$.

The *minimum rank* $\operatorname{mr}(\widehat{G})$ is the smallest possible rank of matrices in $\mathcal{S}(\widehat{G})$, while the *maximum nullity* $M(\widehat{G})$ is the largest possible nullity of matrices in $\mathcal{S}(\widehat{G})$. Similarly, $\operatorname{mr}(\widehat{G}) + M(\widehat{G}) = |V(G)|$. We emphasize that the set $\mathcal{S}(\widehat{G})$ puts no restriction on those entries corresponding to two or more edges, but has additional conditions on diagonal entries from loops, unlike the case in simple graphs.

The *zero forcing process* (for looped multigraphs) is a color-change rule such that:

- a set $B$ of vertices are set blue initially, while others remain white;
- if a vertex $x$, which is not necessarily blue, has exactly one white neighbor $y$, and $e(x,y)=1$, $y$ becomes blue at next step.

If the set $B$ can force all vertices to blue, it is called a *zero forcing set*. The zero forcing number $Z(\widehat{G})$ is defined as the minimum cardinality of a zero forcing set.

**Example 1.** Let $\widehat{G}_1$ and $\widehat{G}_2$ be the looped multigraphs in Figure 1. Then a matrix in $\mathcal{S}(\widehat{G}_1)$ is of the form $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$, so $M(\widehat{G}_1)=1$. And $Z(\widehat{G}_1)=1$, since by setting 1 blue, 2 can force itself.

On the other hand, a matrix in $\mathcal{S}(\widehat{G}_2)$ is of the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$, and so $M(\widehat{G}_2)=0$. Also, since 1 can force 2 in $\widehat{G}_2$ without itself being blue first, the empty set is a zero forcing set for $\widehat{G}_2$ and so $Z(\widehat{G}_2)=0$.

Following the definitions of $\mathrm{mr}(\widehat{G})$ and $Z(\widehat{G})$, there is no difference between $e(i,j)=2$ and $e(i,j)\geq 2$, so we automatically adjust the value $e(i,j)$ to 2 whenever the $e(i,j)\geq 2$. And we just say there is a *double edge* between $i$ and $j$, which includes the case of a double loop.

Next we define some important looped multigraphs from a simple graph. Let $G$ be a simple graph and $I\subseteq V(G)$. Throughout this article, $\widehat{G}_I$ is the looped multigraph obtained from $G$ by adding one loop to each vertex in $I$; the *doubly looped multigraph* $\widehat{G}_{\ell\ell}$ is obtained from $G$ by adding two loops to every vertex. The *enhanced zero forcing number* $\widehat{Z}(G)$ is defined as $\max_{I\subseteq V(G)} Z(\widehat{G}_I)$.

**Observation 2.** For a simple graph $G$, $M(G)=M(\widehat{G}_{\ell\ell})$ and $Z(G)=Z(\widehat{G}_{\ell\ell})$.

**Remark 3.** Let $\widehat{G}$ be a looped multigraph with $e(i,j)=2$ for some not necessarily distinct vertices $i$ and $j$. Let $\widehat{G}_0$, $\widehat{G}_1$ be the looped multigraphs obtained from $\widehat{G}$ by setting $e(i,j)$ to be 0 and 1 respectively. Then

$$M(\widehat{G})=\max\{M(\widehat{G}_0),M(\widehat{G}_1)\} \text{ and}$$
$$Z(\widehat{G})\geq\max\{Z(\widehat{G}_0),Z(\widehat{G}_1)\}.$$

The first equality comes from definition, while the second inequality is because any zero forcing process that can be conducted on $\widehat{G}$ can be applied to $\widehat{G}_0$ or $\widehat{G}_1$.

Proposition 4 provides the relation between $M(\widehat{G})$ and $Z(\widehat{G})$, and the proof is different from the original proof of $M(G)\leq Z(G)$. In a zero forcing process, one white vertex will be forced to blue at each step. We use the notation $x_i \to y_i$ to indicate that at the $i$-th step, $x_i$ forces $y_i$ to blue because $y_i$ is the only white neighbor of $x_i$ and $e(x_i, y_i) = 1$. For each $y_i$, exactly one $x_i$ is chosen to force $y_i$ even if there is more than one choice. The ordered collection of all $x_i \to y_i$ is the *chronological list* of forces defined in [2].

**Proposition 4.** *Let $\widehat{G}$ be a looped multigraph. Then*

$$M(\widehat{G}) \leq Z(\widehat{G}).$$

*Proof.* Let $|V(\widehat{G})| = n$. Let $B$ be a zero forcing set with $|B| = Z(\widehat{G})$, and $\mathcal{F} = (x_i \to y_i : \ i = 1, 2, \ldots, k)$ the corresponding chronological list of forces. Since $\{y_i\}_{i=1}^k$ is the set of all initial white vertices, $k = n - Z(\widehat{G})$.

Let $A$ be a matrix in $\mathcal{S}(\widehat{G})$ with $\operatorname{null}(A) = M(G)$. We permute the rows and columns such that $x_i$ is the $i$-th column and $y_i$ is the $i$-th row. This new matrix can be written as

$$\begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix},$$

where $\widehat{A}_{11}$ is a $k \times k$ submatrix indexed by $\{x_i\}_{i=1}^k$ on columns and $\{y_i\}_{i=1}^k$ on rows. Since $e(x_i, y_i) = 1$ for all $i$, the diagonal entries of $\widehat{A}_{11}$ are nonzero. Also, at the $i$-th step, $\{y_j\}_{j>i}$ are still white, which means there is no edge between $x_i$ and any of $\{y_j\}_{j>i}$. Therefore, $\widehat{A}_{11}$ is of the form

$$\begin{pmatrix} * & & & ? \\ & * & & \\ & & \ddots & \\ O & & & * \end{pmatrix}.$$

So the rank of $A$ is at least $k$, and $M(\widehat{G}) = \operatorname{null}(A) \leq n - k = Z(\widehat{G})$. □

**Corollary 5.** [3] *Let $G$ be a simple graph. Then*

$$M(G) \leq \widehat{Z}(G) \leq Z(G).$$

*Proof.* Let $A$ be a matrix in $\mathcal{S}(G)$ with nullity $M(G)$. Then $A \in \mathcal{S}(\widehat{G}_I)$ for some index set $I$. That means

$$M(G) = M(\widehat{G}_I) \leq Z(\widehat{G}_I) \leq \widehat{Z}(G),$$

by Proposition 4. And the fact $\widehat{Z}(G) \leq Z(G)$ comes from Observation 2 and Remark 3. □

We further define the *addition of two looped multigraphs* as the union of vertex sets and the disjoint union of edge sets, as shown in Figure 1.

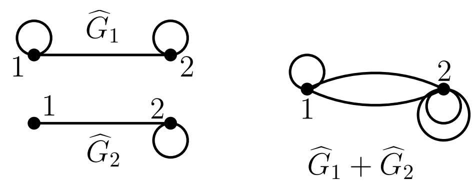


Figure 1: An example of the addition of two looped multigraphs.

**Observation 6.** Let $\widehat{G}_1$ and $\widehat{G}_2$ be two looped multigraphs, and $A \in \mathcal{S}(\widehat{G}_1)$, $B \in \mathcal{S}(\widehat{G}_2)$. Then $A + B \in \mathcal{S}(\widehat{G}_1 + \widehat{G}_2)$.

Therefore the definitions of $\mathcal{S}(\widehat{G})$ and addition of graphs are natural in the sense of graphs and matrices.

Another parameter of interest in the study of minimun rank is the path cover number of simple graphs. For a given simple graph $G$, the *path cover number* $P(G)$ is the minimum number of disjoint induced paths that can cover the vertices of $G$. It is known that $P(G) \leq Z(G)$ [2]. In the case when $G$ is outerplanar, Sinkovic [16] proved that

$$M(G) \leq P(G).$$

However, $M(G)$ and $P(G)$ are in general incomparable [4].

## 2 Sieving Process and Elimination Lemmas

The enhanced zero forcing number gives a powerful upper bound for $M(G)$ between $M(G)$ and $Z(G)$. This fact suggests that a considerable amount of information is hidden in the zero-nonzero patterns of diagonal entries. In fact, we can use $Z(\widehat{G}_I)$ as a tool to analyze the patterns on the diagonal. For a given simple graph $G$ we define

$$\mathcal{I}_k(G) := \{I \subseteq V(G) : \ Z(\widehat{G}_I) \geq k\}.$$

For a matrix $A \in \mathcal{S}(G)$, if $\text{null}(A) \geq k$ then the diagonal pattern of $A$ must fall in $\mathcal{I}_k(G)$. And we may rewrite the enhanced zero forcing number as

$$\widehat{Z}(G) = \max\{k : \ \mathcal{I}_k(G) \neq \varnothing\}.$$

**Example 7.** Let $G$ be the complete graph $K_2$ on two vertices. Then $Z(\widehat{G}_I) = 1$ only when $I = \{1, 2\}$. Therefore,

$$\mathcal{I}_1(G) = \{\{1,2\}\}, \ \mathcal{I}_0(G) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}.$$

That means any matrix $A \in \mathcal{S}(G)$ with nullity 1 must has its diagonal entries nonzero.

This example leads to the following definition.

**Definition 8.** Let $i$ be a vertex of a simple graph $G$. The vertex $i$ is a *zero-vertex* in $\mathcal{I}_k(G)$ if $i$ is not an element of any $I \in \mathcal{I}_k(G)$; it is a *nonzero-vertex* in $\mathcal{I}_k(G)$ if $i$ is an element of every $I \in \mathcal{I}_k(G)$.

This definition gives directly that each vertex in $K_2$ is a nonzero-vertex in $\mathcal{I}_1(K_2)$. In fact, every vertex in a complete graph $K_n$ is a nonzero-vertex in $\mathcal{I}_{n-1}(K_n)$ whenever $n \geq 2$. And the reason for defining zero-vertex and nonzero-vertex is because in so doing we can work on a "template" with more information.

**Definition 9.** Let $G$ be a simple graph and $i \in V(G)$. The looped multigraph $\mathcal{AT}_i(\widehat{G})$ is obtained from $G$ by adding one loop to $i$ and two loops to the others; the looped multigraph $\mathcal{ZT}_i(\widehat{G})$ is obtained from $G$ by adding two loops to all vertices except $i$.

Also, if $N_0 \subseteq V(G)$, then the $\mathcal{DT}_{N_0}(\widehat{G})$ is obtained from $G$ by adding two loops to all vertices beside $N_0$.

**Observation 10.** For a simple graph $G$, if $i$ is a nonzero-vertex in $\mathcal{I}_k(G)$, then any matrix $A \in \mathcal{S}(G)$ with $\mathrm{null}(A) \geq k$ is also a matrix in $\mathcal{S}(\mathcal{AT}_i(\widehat{G}))$; if $i$ is a zero-vertex in $\mathcal{I}_k(G)$, then any matrix $A \in \mathcal{S}(G)$ with $\mathrm{null}(A) \geq k$ is also a matrix in $\mathcal{S}(\mathcal{ZT}_i(\widehat{G}))$; if $N_0$ is a set of some zero-vertices in $\mathcal{I}_k(G)$, then any matrix $A \in \mathcal{S}(G)$ with $\mathrm{null}(A) \geq k$ is also a matrix in $\mathcal{S}(\mathcal{DT}_{N_0}(\widehat{G}))$.

The following example demonstrates a general method for finding a zero-vertex, or a nonzero-vertex.

**Example 11.** Let $G$ be the 5-sun $H_5$ in Figure 2. It is known that $Z(G) = 3$ [1], yet $M(G) = 2$ [4]. We claim vertex 1 is both a zero-vertex and a nonzero-vertex in $\mathcal{I}_3(G)$; this means $\mathcal{I}_3(G) = \varnothing$ and so $M(G) \leq \widehat{Z}(G) \leq 2$.

To see 1 is a zero-vertex in $\mathcal{I}_3(G)$, we suppose that $1 \in I$ for some $I \in \mathcal{I}_3(G)$ and obtain a contradiction. By Remark 3 and the fact $I \in \mathcal{I}_3(G)$, $Z(\mathcal{AT}_1(\widehat{G})) \geq Z(\widehat{G}_I) \geq 3$. But $B_0 := \{3, 5\}$ is a zero forcing set of $\mathcal{AT}_1(\widehat{G})$, so $Z(\mathcal{AT}_1(\widehat{G})) \leq 2$. This violates our assumption, so 1 is a zero-vertex.

Similarly, if $1 \notin I$ for some $I \in \mathcal{I}_3(G)$, $Z(\mathcal{ZT}_1(\widehat{G})) \geq Z(\widehat{G}_I) \geq 3$ by Remark 3. Then $B_0 = \{3, 9\}$ is a zero forcing set of $\mathcal{ZT}_1(\widehat{G})$, so the fact $Z(\mathcal{ZT}_1(\widehat{G})) \leq 2$ indicates 1 is also a nonzero-vertex.

As in the example, every vertex subset $B$ of size $k - 1$ serves as a *sieve* for $\mathcal{I}_k(G)$: if $B$ is a zero forcing set for $\widehat{G}_I$, then $I$ cannot be an element in $\mathcal{I}_k(G)$. Through this *sieving process*, some zero-nonzero patterns are guaranteed in $\mathcal{I}_k(G)$. Consequently, some submatrices are guaranteed invertible, and so the Schur complement can be applied.

**Definition 12.** Let $A$ be an $n \times n$ matrix with rows and columns indexed by $V$ and $\alpha, \beta \subseteq V$. Then $A[\alpha, \beta]$ is defined as the submatrix induced on those rows in $\alpha$ and columns in $\beta$. For convention, we write $A[\alpha] := A[\alpha, \alpha]$, $A(\alpha) := A[\overline{\alpha}, \overline{\alpha}]$, $A(\alpha, \beta] := A[\overline{\alpha}, \beta]$, and $A[\beta, \alpha) := A[\beta, \overline{\alpha}]$, where $\overline{\alpha}$ is the set $V \setminus \alpha$. If $A[\alpha]$ is invertible, then the matrix $A(\alpha) - A(\alpha, \alpha]A[\alpha]^{-1}A[\alpha, \alpha)$ is called the *Schur complement* of $A$ on $\alpha$.

**Remark 13.** A matrix $A$ and its Schur complement have the same nullity [14].

Since the maximum nullity of a simple graph is still of our greatest interest, we focus on the looped graphs instead of looped multigraphs. That is, loops and double loops are allowed, so that we can analyze the diagonal patterns. As a result, Definition 14, Lemma 15, and Theorem 16, 18, 21 all start with looped graphs, but possibly end with looped multigraphs, as a result of operations that render some entries unknown.

**Definition 14.** Let $\widehat{G}$ be a looped graph with $i \in V(\widehat{G})$. Define $\mathcal{AE}_i(\widehat{G})$ to be the looped multigraph on the vertex set $V(\widehat{G}) \setminus \{i\}$ with $e(x, y) = 1$ for all $x, y \in N(i) \setminus \{i\}$ and $e(i, j) = 0$ otherwise, where $x, y$ are not necessarily distinct.

If, in addition, $j$ is a vertex other than $i$ with $e(i, j) = 1$ in $\widehat{G}$, define $N_i := N(i) \setminus \{i, j\}$ and $N_j := N(j) \setminus \{i, j\}$, and $\mathcal{ZE}_i^j(G)$ to be the looped multigraph on the vertex set $V(\widehat{G}) \setminus \{i, j\}$ with

- $e(x, y) = 1$ for $x \in N_i$, $y \in N_j \setminus N_i$;
- $e(x, y) = 2$ for $x, y \in N_i$;
- $e(x, y) = 0$ otherwise,

where $x, y$ are not necessarily distinct.

Also, define $\mathcal{DE}_i^j(G)$ to be the looped multigraph defined on the vertex set $V(\widehat{G}) \setminus \{i, j\}$ with

- $e(x, y) = 1$ for $x, y \in N_i \cup N_j$ but $x, y$ are not both in one of $N_i \setminus N_j$, $N_j \setminus N_i$, or $N_i \cap N_j$;
- $e(x, y) = 2$ for distinct $x, y \in N_i \cap N_j$;
- $e(x) = 1$ for $x \in N_i \cap N_j$.

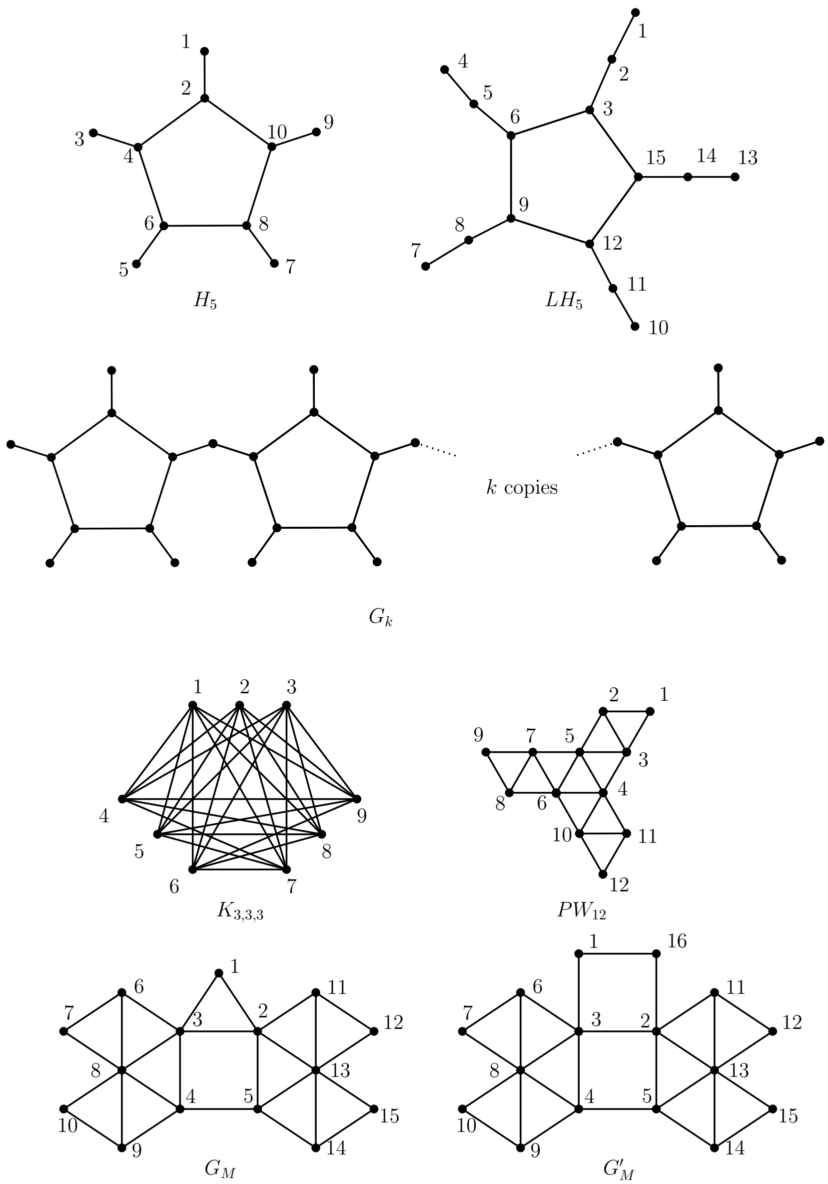


Figure 2: A collection of graphs $G$ with $M(G) < Z(G)$.

- $e(x,y) = 0$ otherwise, where $x, y$ are not necessarily distinct.

**Lemma 15.** *Let $\widehat{G}$ be a looped graph and $A \in \mathcal{S}(\widehat{G})$. Let $\alpha \subseteq V(G)$ be an index set such that $A[\alpha]$ is invertible and $C \coloneqq A(\alpha,\alpha]A[\alpha]^{-1}A[\alpha,\alpha)$.*

**(1)** *If $\alpha = \{i\} \subseteq V(\widehat{G})$ and $e(i) = 1$, then $C \in \mathcal{S}(\mathcal{AE}_i(\widehat{G}))$.*

**(2)** *If $\alpha = \{i,j\} \subseteq V(\widehat{G})$, $e(i) = 0$, and $e(i,j) = 1$, then $C \in \mathcal{S}(\mathcal{ZE}_i^j(\widehat{G}))$.*

**(3)** *If $\alpha = \{i,j\} \subseteq V(\widehat{G})$, $e(i) = e(j) = 0$, and $e(i,j) = 1$, then $C \in \mathcal{S}(\mathcal{DE}_i^j(\widehat{G}))$.*

*Proof.* For **(1)**, since $e(i) = 1$ in $\widehat{G}$, the $i,i$-entry of $A$ must be nonzero, so $C$ is defined. Each vertex in $N(i) \setminus \{i\}$ represents a nonzero entry in $A(\alpha,\alpha]$, or $A[\alpha,\alpha)$. Therefore $C \coloneqq A(\alpha,\alpha]A[\alpha]^{-1}A[\alpha,\alpha)$ is a matrix with the zero-nonzero pattern described by $\mathcal{AE}_i(\widehat{G})$.

For **(2)**, since $e(i) = 0$ and $e(i,j) = 1$ in $\widehat{G}$, the matrix $A[\alpha]$ is of the form

$$\begin{pmatrix} 0 & * \\ * & ? \end{pmatrix},$$

which is guaranteed invertible with $A[\alpha]^{-1}$ of the form

$$\begin{pmatrix} ? & * \\ * & 0 \end{pmatrix}.$$

In $A(\alpha,\alpha]$, saying $i \le j$, each vertex in $N_i \cap N_j$ represents a row of the form $\begin{pmatrix} * & * \end{pmatrix}$; each vertex in $N_i \setminus N_j$ represents a row of the form $\begin{pmatrix} * & 0 \end{pmatrix}$; each vertex in $N_j \setminus N_i$ represents a row of the form $\begin{pmatrix} 0 & * \end{pmatrix}$; and each vertiex not in $N_i \cup N_j$ represents a row of the form $\begin{pmatrix} 0 & 0 \end{pmatrix}$. By doing the symbolic matrix operation

$$\begin{pmatrix} * & * \\ * & 0 \\ 0 & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} ? & * \\ * & 0 \end{pmatrix} \begin{pmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \end{pmatrix} \sim \begin{pmatrix} ? & ? & * & 0 \\ ? & ? & * & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the zero-nonzero pattern of $C$ agrees with that given by $\mathcal{ZE}_i^j(\widehat{G})$.

For **(3)**, the conditions imply the matrix $A[\alpha]$ is of the form

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

which is guaranteed invertible with $A[\alpha]^{-1}$ of the form

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Similarly, the symbolic matrix operation is

$$\begin{pmatrix} * & * \\ * & 0 \\ 0 & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \begin{pmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \end{pmatrix} \sim \begin{pmatrix} ? & * & * & 0 \\ * & 0 & * & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, if $x$ is a vertex in $N_i \cap N_j$, then the $x, x$-entry in $C$ is nonzero since the $i, j$-entry and the $j, i$-entry are the same in $A$. □

**Theorem 16** (Nonzero Elimination Lemma)**.** *Let $\widehat{G}$ be a looped graph with $i \in V(\widehat{G})$. Suppose $e(i) = 1$ in $\widehat{G}$. If $A \in \mathcal{S}(\widehat{G})$, then* $\operatorname{null}(A) \le M(\widehat{G} \setminus \{i\} + \mathcal{AE}_i(\widehat{G}))$.

*In particular, on a simple graph $G$, if $i$ is a nonzero-vertex in $\mathcal{I}_k(G)$ and $M(G) \ge k$, then $k \le M(\mathcal{AT}_i(\widehat{G}) \setminus \{i\} + \mathcal{AE}_i(\widehat{G}))$.*

*Proof.* Suppose $A \in \mathcal{S}(\widehat{G})$. Let $\alpha = \{i\}$ and $C := A(\alpha, \alpha]A[\alpha]^{-1}A[\alpha, \alpha)$. Then $A(\alpha) \in \mathcal{S}(\widehat{G} \setminus \{i\})$ and $C \in \mathcal{S}(\mathcal{AE}_i(\widehat{G}))$ by Lemma 15. Consequently, the Schur complement $A(\alpha) - C$ is in $\mathcal{S}(\widehat{G} \setminus \{i\} + \mathcal{AE}_i(\widehat{G}))$, and so

$$\operatorname{null}(A) = \operatorname{null}(A(\alpha) - C) \le M(\widehat{G} \setminus \{i\} + \mathcal{AE}_i(\widehat{G})).$$

For the case of a simple graph, if $i$ is a nonzero-vertex in $\mathcal{I}_k(G)$ and $\operatorname{null}(A) \ge k$ for some $A \in \mathcal{S}(G)$, then $A \in \mathcal{S}(\mathcal{AT}_i(\widehat{G}))$ by Observation 10. Therefore, $k \le M(\mathcal{AT}_i(\widehat{G}) \setminus \{i\} + \mathcal{AE}_i(\widehat{G}))$. □

**Example 17.** Let $G$ be the simple graph $LH_5$ in Figure 2. Since there are 5 vertices of degree 1, $Z(G) \ge P(G) \ge 3$. We know $Z(G) = 3$, because $\{1, 4, 7\}$ is a zero forcing set, and $M(G) = 2$ by the cut-vertex reduction formula [4]. Furthermore, by considering $I = \{1, 2, \ldots, 15\}$, $Z(\widehat{G}_I) = 3$ and so $\widehat{Z}(G) = 3$. Besides, a similar argument to Example 11 suggests the vertex 2 is both a zero-vertex and a nonzero-vertex in $\mathcal{I}_3(G \setminus \{1\})$, so $\widehat{Z}(G \setminus \{1\}) = 2$.

Now we choose $B = \{4, 7\}$ as our first sieve. If 1 is not in some $I \in \mathcal{I}_3(G)$, then 1 can force 2 to blue, and $B$ is a zero forcing set of size 2, which is impossible. Therefore, 1 is a nonzero-vertex in $\mathcal{I}_3(G)$.

Suppose $M(G) = 3$ and apply Theorem 16. The looped multigraph $\mathcal{AE}_1(\widehat{G})$ is a bunch of isolated vertices with one loop on vertex 2, and $\mathcal{AT}_1(\widehat{G}) \setminus \{1\}$ is the doubly looped multigraph of $G \setminus \{1\}$. So $\mathcal{S}(\mathcal{AT}_1(\widehat{G}) \setminus$

$\{1\} + \mathcal{AE}_1(\widehat{G})) = \mathcal{S}(H)$, where $H$ is the simple graph $G - 1$. By Theorem 16,

$$3 \le M(H) \le \widehat{Z}(H) = 2,$$

a contradiction. As a consequence, we know $M(G) \le 2$.

On the other hand, similar process can be done on a zero-vertex.

**Theorem 18** (Zero Elimination Lemma). *Let $\widehat{G}$ be a looped graph with distinct $i, j \in V(\widehat{G})$. Suppose $e(i) = 0$ and $e(i,j) = 1$ in $\widehat{G}$. If $A \in \mathcal{S}(\widehat{G})$, then* $\operatorname{null}(A) \le M(\widehat{G} \setminus \{i\} + \mathcal{ZE}_i^j(\widehat{G}))$.

*In particular, on a simple graph $G$ with distinct $i, j \in V(G)$, if $i$ is a zero-vertex in $\mathcal{I}_k(G)$, $e(i,j) = 1$, and $M(G) \ge k$, then $k \le M(\mathcal{ZT}_i(\widehat{G}) \setminus \{i,j\} + \mathcal{ZE}_i^j(\widehat{G}))$.*

*Proof.* Suppose $A \in \mathcal{S}(\widehat{G})$. Let $\alpha = \{i,j\}$ and $C := A(\alpha,\alpha]A[\alpha]^{-1}A[\alpha,\alpha)$. Then $A(\alpha) \in \mathcal{S}(\widehat{G} \setminus \{i,j\})$ and $C \in \mathcal{S}(\mathcal{ZE}_i^j(\widehat{G}))$ by Lemma 15. Consequently, the Schur complement $A(\alpha) - C$ is in $\mathcal{S}(\widehat{G} \setminus \{i,j\} + \mathcal{ZE}_i^j(\widehat{G}))$, and so

$$\operatorname{null}(A) = \operatorname{null}(A(\alpha) - C) \le M(\widehat{G} \setminus \{i,j\} + \mathcal{ZE}_i^j(\widehat{G})).$$

For the case of a simple graph, if $i$ is a zero-vertex in $\mathcal{I}_k(G)$, $e(i,j) = 1$, and $\operatorname{null}(A) \ge k$ for some $A \in \mathcal{S}(G)$, then $A \in \mathcal{S}(\mathcal{ZT}_i(\widehat{G}))$ by Observation 10. Therefore, $k \le M(\mathcal{ZT}_i(\widehat{G}) \setminus \{i,j\} + \mathcal{ZE}_i^j(\widehat{G}))$. □

**Example 19.** Let $G$ be the simple graph $G_M$ in Figure 2. We have $Z(G) = 5$ by use of the *Sage* minimum rank software [8].

Setting $B = \{8, 9, 13, 14\}$ as a sieve, $B$ is always a zero forcing set of $\widehat{G}_I$ whenever $1 \in I$, and so 1 is a zero-vertex in $\mathcal{I}_5(G)$. Suppose $M(G) = 5$ and apply Theorem 18 on vertices 1 and 2. Then $N_1 = \{3\}$ and $N_2 = \{3, 7, 13, 15\}$. So the looped multigraph $\mathcal{ZE}_1^2(\widehat{G})$ has edges $e(3,7) = e(3,13) = e(3,15) = 1$ and $e(3) = 2$, and $\mathcal{ZT}_1(\widehat{G}) \setminus \{1,2\}$ is the doubly looped multigraph of $G \setminus \{1,2\}$. Therefore, $\mathcal{S}(\mathcal{ZT}_1(\widehat{G}) \setminus \{1,2\} + \mathcal{ZE}_1^2(\widehat{G})) = \mathcal{S}(H)$, where $H$ is the simple graph in Figure 3. By Theorem 18,

$$5 \le M(H) \le Z(H) \le 4,$$

a contradiction. Hence $M(G) \le 4$, and actually $M(G) = 4$ by the reduction formula [13].

**Theorem 20** (Simple Elimination Lemma). *Let $G$ be a simple graph with distinct $i, j \in V(G)$ and $e(i,j) = 1$. Then*

$$M(G) \le \max\{M(\mathcal{AT}_i(\widehat{G}) \setminus \{i\} + \mathcal{AE}_i(\widehat{G})), M(\mathcal{ZT}_i(\widehat{G}) \setminus \{i,j\} + \mathcal{ZE}_i^j(\widehat{G}))\}.$$

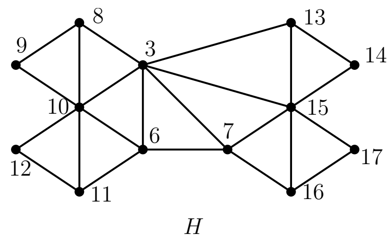


Figure 3: The underlying graph of $\mathcal{ZT}_1(\widehat{G}) \setminus \{1,2\} + \mathcal{ZE}_1^2(\widehat{G})$.

*Proof.* Let $A \in \mathcal{S}(G)$ be a matrix with nullity $M(G)$. Then the $i,i$-entry is either zero or nonzero. If it is nonzero, then $A \in \mathcal{S}(\mathcal{AT}_i(\widehat{G}))$, and so $\text{null}(A) \leq M(\mathcal{AT}_i(\widehat{G}) \setminus \{i\} + \mathcal{AE}_i(\widehat{G}))$ by Theorem 16; if it is zero, then $A \in \mathcal{S}(\mathcal{ZT}_i(\widehat{G}))$, and so $\text{null}(A) \leq M(\mathcal{ZT}_i(\widehat{G}) \setminus \{i,j\} + \mathcal{ZE}_i^j(\widehat{G}))$ by Theorem 18. Consequently,

$$M(G) = \text{null}(A)$$
$$\leq \max\{M(\mathcal{AT}_i(\widehat{G}) \setminus \{i\} + \mathcal{AE}_i(\widehat{G})), M(\mathcal{ZT}_i(\widehat{G}) \setminus \{i,j\} + \mathcal{ZE}_i^j(\widehat{G}))\}.$$

□

**Theorem 21** (Double Zero Elimination Lemma)**.** *Let $\widehat{G}$ be a looped graph with distinct $i,j \in V(\widehat{G})$. Suppose $e(i) = e(j) = 0$ and $e(i,j) = 1$. If $A \in \mathcal{S}(\widehat{G})$, then* $\text{null}(A) \leq M(\widehat{G} \setminus \{i,j\} + \mathcal{DE}_i^j(\widehat{G}))$.

*In particular, on a simple graph $G$, if $N_0$ is a set of some zero-vertices in $\mathcal{I}_k(G)$ with distinct $i,j \in N_0$, $e(i,j) = 1$, and $M(G) \geq k$, then $k \leq M(\mathcal{DT}_{N_0}(\widehat{G}) \setminus \{i,j\} + \mathcal{DE}_i^j(\widehat{G}))$.*

*Proof.* Suppose $A \in \mathcal{S}(\widehat{G})$. Let $\alpha = \{i,j\}$ and $C \coloneqq A(\alpha,\alpha]A[\alpha]^{-1}A[\alpha,\alpha)$. Then $A(\alpha) \in \mathcal{S}(\widehat{G} \setminus \{i,j\})$ and $C \in \mathcal{S}(\mathcal{DE}_i^j(\widehat{G}))$ by Lemma 15. Consequently, the Schur complement $A(\alpha) - C$ is in $\mathcal{S}(\widehat{G} \setminus \{i,j\} + \mathcal{DE}_i^j(\widehat{G}))$, and so

$$\text{null}(A) = \text{null}(A(\alpha) - C) \leq M(\widehat{G} \setminus \{i,j\} + \mathcal{DE}_i^j(\widehat{G})).$$

For the case of a simple graph, if $N_0$ is a set of some zero-vertices in $\mathcal{I}_k(G)$ with distinct $i,j \in N_0$, $e(i,j) = 1$, and $\text{null}(A) \geq k$ for some $A \in \mathcal{S}(G)$, then $A \in \mathcal{S}(\mathcal{DT}_{N_0}(\widehat{G}))$ by Observation 10. Therefore, $k \leq M(\mathcal{DT}_{N_0}(\widehat{G}) \setminus \{i,j\} + \mathcal{DE}_i^j(\widehat{G}))$. □

**Example 22.** Let $G$ be the simple graph $K_{3,3,3}$ in Figure 2. It is known $Z(G) = 7$ [1] and $M(G) = 6$ by [6].

By setting $B = \{3,4,5,6,7,8\}$ as a sieve, vertex 1 is a zero-vertex in $\mathcal{I}_7(G)$. Similar argument ensures that each vertex is a zero-vertex in $\mathcal{I}_7(G)$. Suppose $M(G) = 7$ and apply Theorem 21 to the vertices 1 and 4. The looped multigraph $\mathcal{DT}_{N_0}(\widehat{G}) \smallsetminus \{1,4\} + \mathcal{DE}_i^j(\widehat{G})$ has 7 vertices and three of them have exactly one loop on each of them. That means $7 \leq M(\mathcal{DT}_{N_0}(\widehat{G}) \smallsetminus \{1,4\} + \mathcal{DE}_i^j(\widehat{G})) < 7$, a contradiction. Hence $M(G) \leq 6$.

**Note.** The third statement in Lemma 15 relies on the symmetry of matrices and the fact $a + a \neq 0$ whenever $a$ is a nonzero real number. Therefore, Theorem 21 fails when the considered field is of characteristic 2, or the considered matrices are not necessarily symmetric. In fact, the maximum nullity of $K_{3,3,3}$ is 7 when the considered field is $\mathbb{Z}_2$ [11].

# 3 Conclusion

| | $M$ | $Z$ | $\widehat{Z}$ | $SP$ | $P$ |
|---|---|---|---|---|---|
| $H_5$ | 2 | 3 | 2 | 2 | 3 |
| $LH_5$ | 2 | 3 | 3 | 2 | 3 |
| $G_k$ | $k+1$ | $2k+1$ | $k+1$ | $k+1$ | $2k+1$ |
| $K_{3,3,3}$ | 6 | 7 | 7 | 6 | 3 |
| $PW_{12}$ | 3 | 4 | 4 | 3 | 3 |
| $G_M$ | 4 | 5 | 4 | 4 | 5 |
| $G'_M$ | 4 | 5 | 5 | 4 | 5 |

Table 1: $M(G)$ and its upper bounds.

The sieving process and the elimination lemmas provide a new way to find the upper bounds for $M(G)$. In fact, many known graphs $G$ with $M(G) < Z(G)$ can be explained by the sieving process and the elimination lemmas. Figure 2 illustrates a collection of graphs whose maximum nullity and zero forcing number are different; Table 1 provides their maximum nullity $M$ and many upper bounds, including the zero forcing number $Z$, the enhanced zero forcing number $\widehat{Z}$, and finally the upper bound $SP$ given by the sieving process. The path cover number is also listed, though $K_{3,3,3}$ is not outerplanar.

The values in Table 1 were established in examples in Section 2, except the values of the parameters of $G_k$, $PW_{12}$ and $G'_M$ can be found in [5, 2, 9] respectively, or they can be obtained by software [8]; $Z(G)$, $P(G)$, and $\widehat{Z}(G)$ are computed by either systematical arguments or brute force; while detailed discussions of the bounds from sieving processes can be found in [9].

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# References

[1] AIM minimum rank-special graphs work group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications* **428** (2008) 1628–1648.

[2] F. Barioli, W. Barrett, S.M. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications* **433** (2010) 401–411.

[3] F. Barioli, W. Barrett, S.M. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. *Journal of Graph Theory* **72** (2013) 146–177.

[4] F. Barioli, S. Fallat, and L. Hogben. Computation of minimal rank and path cover number for certain graphs. *Linear Algebra and its Applications* **392** (2004) 289–303.

[5] F. Barioli, S. Fallat, and L. Hogben. On the difference between the maximum multiplicity and path cover number for tree-like graphs. *Linear Algebra and its Applications* **409** (2005) 13–31.

[6] W. Barrett, H. van der Holst, and R. Loewy. Graphs whose minimal rank is two. *Electronic Journal of Linear Algebra* **11** (2004) 258–280.

[7] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Physical Review Letters* **99**, (2007) 100501.

[8] S. Butler, L. DeLoss, J. Grout, H.T. Hall, J. LaGrange, T. McKay, J. Smith, and G. Tims. Minimum Rank Library (*Sage* programs for calculating bounds on the minimum rank of a graph, and for computing zero forcing parameters). Available at `http://sage.cs.drake.edu/home/pub/67/`. For more information contact Jason Grout at `jason.grout@drake.edu`.

[9] C.-H. Lin. Applications of zero forcing number to the minimum rank problem. *Master Thesis*, National Taiwan University (2011).

[10] L. DeLoss, J. Grout, L. Hogben, T. McKay, J. Smith, and G. Tims. Techniques for determining the minimum rank of a small graph. *Linear Algebra and its Applications* **432** (2010) 2995–3001.

[11] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications* **426** (2007) 558–582.

[12] H.T. Hall, L. Hogben, R. Martin, and B. Shader. Expected values of parameters associated with the minimum rank of a graph. *Linear Algebra and its Applications* **433** (2010) 101–117.

[13] H. van der Holst. The maximum corank of graphs with a 2-separation. *Linear Algebra and its Applications* **428** (2008) 1587–1600.

[14] R.A. Horn and C.R. Johnson. *Matrix Analysis*, 2nd Ed. Cambridge University Press (c) 2013.

[15] R. Martin, Personal communication.

[16] J. Sinkovic. Maximum nullity of outer planar graphs and the path cover number. *Linear Algebra and its Applications* **432** (2010) 2052–2060.