Inverse eigenvalue problem of a graph

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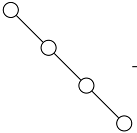
Inverse eigenvalue problem of a graph (IEP-G)

Let G be a graph. Define $\mathcal{S}(G)$ as the family of all real symmetric matrices $A = \left[a_{ij}\right]$ such that

$$a_{ij} \begin{cases} \neq 0 & \text{if } ij \in E(G), i \neq j; \\ = 0 & \text{if } ij \notin E(G), i \neq j; \\ \in \mathbb{R} & \text{if } i = j. \end{cases}$$



$$\longrightarrow \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \underset{?}{\overset{\text{spec}}{\rightleftarrows}} \{1,2,3,4\}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Inverse eigenvalue problem of a graph (IEP-G)

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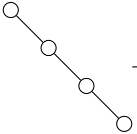

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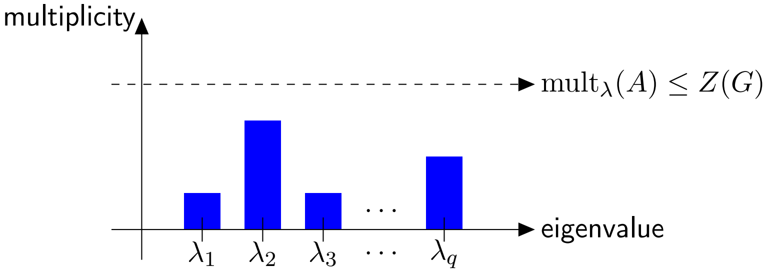
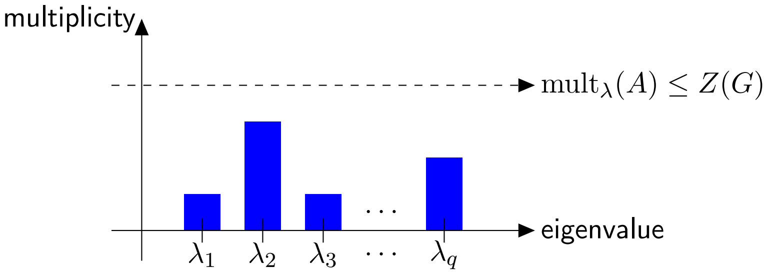
IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?



$$\mathrm{spec}(A) = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\} \implies \begin{array}{l} \mathbf{m}(A) = (m_1, \ldots, m_q), \\ q(A) = q \end{array}$$



Questions

What are possible $\mathbf{m}(A)$ and what are

$$M(G) = \max\{\text{mult}_\lambda(A) : A \in \mathcal{S}(G), \lambda \in \text{spec}(A)\},$$
$$q(G) = \min\{q(A) : A \in \mathcal{S}(G)\}?$$

IEP-P_n

$$\begin{bmatrix} ? & * & & & \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ & & & * & ? \end{bmatrix} \in \mathcal{S}(P_n) \qquad \lambda_1 < \cdots < \lambda_n$$

- $\{$Rows $2 \sim n\}$ and $\{$Rows $1 \sim n-1\}$ are always independent.
- $\mathrm{mult}(\lambda) = \mathrm{null}(A - \lambda I) \leq 1$ for any $A \in \mathcal{S}(P_n)$ and $\lambda \in \mathbb{R}$.

IEP-P_n

$$\begin{bmatrix} ? & * & & & \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ & & & * & ? \end{bmatrix} \in \mathcal{S}(P_n) \qquad \lambda_1 < \cdots < \lambda_n$$

Theorem (Gray and Wilson 1976; and Hald 1976)

For any set $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of n distinct real numbers, there is a matrix $A \in \mathcal{S}(P_n)$ such that $\operatorname{spec}(A) = \Lambda$.

IEP-C_n

$$\begin{bmatrix} ? & * & & & * \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ * & & & * & ? \end{bmatrix} \in \mathcal{S}(C_n)$$

$\lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots$, or

$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \cdots$

For example, $(2, 1, 2)$ is not possible for C_5.

- $\{$Rows $2 \sim n-1\}$ is always independent.
- $\mathrm{mult}(\lambda) = \mathrm{null}(A - \lambda I) \leq 2$ for any $A \in \mathcal{S}(C_n)$ and $\lambda \in \mathbb{R}$.

IEP-C_n

$$\begin{bmatrix} ? & * & & & * \\ * & ? & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & & * \\ * & & & * & ? \end{bmatrix} \in \mathcal{S}(C_n)$$

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For example, $(2, 1, 2)$ is not possible for C_5.

Theorem (Ferguson 1980)

For any set $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of n real numbers satisfying one of the conditions above, there is a matrix $A \in \mathcal{S}(C_n)$ such that $\operatorname{spec}(A) = \Lambda$.

Signature similarity

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0 & 0\\2 & 3 & -4 & 0\\0 & -4 & 5 & -6\\0 & 0 & -6 & 7\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Definition

A signature matrix is a matrix whose diagonal entries are 1 or -1. Two matrices A and B are signature similar if $B = DAD$ for some signature matrix.

Observation

Every matrix $A \in \mathcal{S}(P_n)$ is signature similar to a matrix $A' \in \mathcal{S}(P_n)$ whose off-diagonal entries are nonnegative.

What do we know about the eigenvectors?

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 6 & 7 \end{bmatrix} = QDQ^\top \text{ with } Q = \begin{bmatrix} 0.33 & 0.86 & 0.38 & 0.05 \\ -0.58 & -0.13 & 0.75 & 0.28 \\ 0.63 & -0.36 & 0.17 & 0.67 \\ -0.4 & 0.34 & -0.5 & 0.69 \end{bmatrix}$$

Theorem (Ferguson 1980)

Let $A \in \mathcal{S}(P_n)$ with nonnegative off-diagonal entries. Suppose $\lambda_1 < \cdots < \lambda_n$ are the eigenvalues of A and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ the corresponding orthonormal eigenbasis. Then $(\mathbf{v}_i)_1(\mathbf{v}_i)_n$ is sign alternating for $i = 1, \ldots, n$.

Adjugate

Let A be an $n \times n$ matrix.

- $A(i,j)$ is the submatrix of A by removing the i-th row and the j-th column.
- The i,j-cofactor of A is

$$c_{i,j}(A) = (-1)^{i+j}\det(A(i,j)).$$

- The adjugate of A is $A^{\mathrm{adj}} = \begin{bmatrix} c_{i,j} \end{bmatrix}^\top$.
- It is known that $AA^{\mathrm{adj}} = A^{\mathrm{adj}}A = \det(A)I$, so

$$A^{\mathrm{adj}} = \begin{cases} \det(A)A^{-1} & \text{if } \operatorname{rank}(A) = n, \\ O & \text{if } \operatorname{rank}(A) \leq n-2. \end{cases}$$

Eigenvector-eigenvalue identity

Theorem (Eigenvector-eigenvalue identity)

Let A *be an* $n \times n$ *real symmetric matrix. Suppose* $\lambda_1 \leq \cdots \leq \lambda_n$ *are the eigenvalues of* A *and* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *the corresponding orthonormal eigenbasis. If* $\mathrm{mult}(\lambda_i) = 1$*, then*

$$(A - \lambda_i I)^{\mathrm{adj}} = \left(\prod_{j \neq i} (\lambda_j - \lambda_i)\right) \mathbf{v}_i \mathbf{v}_i^\top.$$

Corollary

When $A \in \mathcal{S}(P_n)$ *is a matrix with nonnegative off-diagonal entries,*

$$\mathrm{sgn}((\mathbf{v}_i)_1 (\mathbf{v}_i)_n) = (-1)^{1+n} \det((A - \lambda_i)(n,1)) \prod_{j \neq i} (\lambda_j - \lambda_i).$$

Theorem (Ferguson 1980)

Let $A \in \mathcal{S}(C_n)$ with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. If $\lambda_i = \lambda_{i+1}$ and $\lambda_j = \lambda_{j+1}$ for some $i < j$, then $j - i$ is even.

Sketch of the proof

By signature similarity, we may assume

$$A = \begin{bmatrix} A(n) & \mathbf{b} \\ \mathbf{b}^\top & c \end{bmatrix} \text{ with } \mathbf{b} = \begin{bmatrix} x \\ \mathbf{0} \\ y \end{bmatrix}$$

such that $A(n) \in \mathcal{S}(P_{n-1})$ with nonnegative off-diagonal entries and eigenvalues $\mu_1 < \cdots < \mu_{n-1}$.
(Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ be the corresponding orthonormal eigenbasis of $A(n)$.)

Spectrum of $A \in \mathcal{S}(C_n)$

Theorem (Ferguson 1980)

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Sketch of the proof

By the Cauchy interlacing theorem,

$$\lambda_i \leq \mu_i \leq \lambda_{i+1} = \lambda_i$$

implies $\lambda_i = \mu_i = \lambda_{i+1}$ and similarly $\lambda_j = \mu_j = \lambda_{j+1}$.

Spectrum of $A \in \mathcal{S}(C_n)$

Theorem (Ferguson 1980)

Let $A \in \mathcal{S}(C_n)$ with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. If $\lambda_i = \lambda_{i+1}$ and $\lambda_j = \lambda_{j+1}$ for some $i < j$, then $j - i$ is even.

Sketch of the proof

$$A = \begin{bmatrix} A(n) & \mathbf{b} \\ \mathbf{b}^\top & c \end{bmatrix} \text{ with } \mathbf{b} = \begin{bmatrix} x \\ \mathbf{0} \\ y \end{bmatrix}$$

- Since $\operatorname{mult}_A(\lambda_i) = 2$ and $\operatorname{mult}_{A(n)}(\lambda_i) = 1$, $\mathbf{b} \in \operatorname{Col}(A(n) - \mu_i I)$.
- $\mathbf{b} \perp \mathbf{v}_i \implies \operatorname{sgn}(xy) = -\operatorname{sgn}((\mathbf{v}_i)_1)(\mathbf{v}_i)_{n-1})$ (same for j)
- Since $\operatorname{sgn}(xy)$ is fixed and $\operatorname{sgn}((\mathbf{v}_j)_1)(\mathbf{v}_j)_{n-1})$ is sign alternating, $j - i$ is even.

For the IEP-G, we need ...

- Combinatorial tools: zero forcing, unique shortest path, variants of zero forcing ...
- Theory of symmetric matrices: Cauchy interlacing theorem, eigenvector-eigenvalue identity, Rayleigh quotient, Parter–Wiener theorem, Godsil's lemma, ...
- Analytic tools: implicit function theorem, inverse function theorem, ...

An introductory article

S. M. Fallat, L. Hogben, J. C.-H. Lin, and B. Shader.
The inverse eigenvalue problem of a graph,
zero forcing, and related parameters.
Notices Amer. Math. Soc., 67:257–261, February, 2020.

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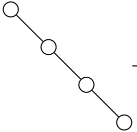

$$\longrightarrow \begin{tikzcd} \begin{bmatrix} ? & * & 0 & 0 \\ * & ? & * & 0 \\ 0 & * & ? & * \\ 0 & 0 & * & ? \end{bmatrix} \arrow[r, bend left, "\text{spec}"] & \{1,2,3,4\} \arrow[l, bend left, "?"] \end{tikzcd}$$

IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

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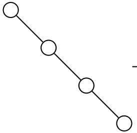

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IEP-G: What are the possible spectra of a matrix in $\mathcal{S}(G)$?

Supergraph lemma

Lemma (BFHHLS 2017)

Let G and H' be two graphs with $V(G) = V(H')$ and $E(G) \subseteq E(H')$. If $A \in \mathcal{S}(G)$ has the SSP, then there is a matrix $A' \in \mathcal{S}(H')$ such that

- $\operatorname{spec}(A') = \operatorname{spec}(A)$,
- *A' has the SSP, and*
- *$\|A' - A\|$ can be chosen arbitrarily small.*

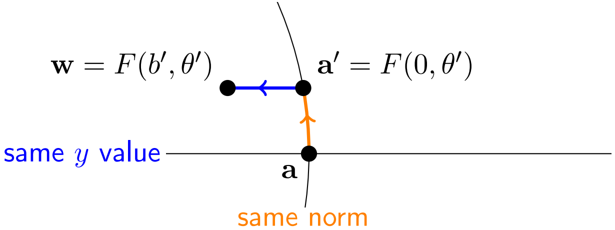
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

SSP will be defined later

Inverse function theorem in $\mathbb{R}^2$

Fix a point $\mathbf{a} \in \mathbb{R}^2$. Combine two perturbations:

$$\begin{cases} \mathbf{a} + b\mathbf{e}_1 & \text{(same } y\text{)} \\ R_\theta \mathbf{a} & \text{(same norm)} \end{cases} \implies F(b, \theta) = R_\theta \mathbf{a} + b\mathbf{e}_1$$



$\frac{dF}{db,\theta}$ invertible $\implies$ any nearby $\mathbf{w}$ can be written as $\mathbf{w} = F(b', \theta')$

For whatever y value nearby, there is $\mathbf{a}'$ with $\|\mathbf{a}'\| = \|\mathbf{a}\|$.

Inverse function theorem

Theorem (Inverse function theorem)

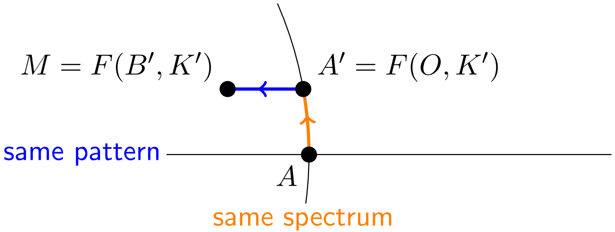
Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathbf{u}_0 \in U$ is invertible, then F is locally invertible around $\mathbf{u}_0$.

Theorem (FHLS 2022)

Let $F : U \to W$ be a smooth function. If $\dot{F}$ at a point $\mathbf{u}_0 \in U$ is surjective, then F is locally surjective around $\mathbf{u}_0$.

Fix a point $A \in \mathcal{S}(G)$. Combine two perturbations:

$$\begin{cases} A + B & \text{(same pattern)} \\ e^{-K}Ae^{K} & \text{(same spectrum)} \end{cases} \implies F(B,K) = e^{-K}Ae^{K} + B$$



$\dot{F}$ surjective $\implies$ any nearby M can be written as $M = F(B', K')$

For whatever pattern nearby, there is A' with $\mathrm{spec}(A') = \mathrm{spec}(A)$.

Pattern perturbation

$$F(B, K) = e^{-K} A e^{K} + B$$

Define $\mathcal{S}^{\mathrm{cl}}(G)$ as the topological closure of $\mathcal{S}(G)$:

$$\mathcal{S}^{\mathrm{cl}}(G) = \{A = \begin{bmatrix} a_{i,j} \end{bmatrix} \in \mathrm{Sym}_n(\mathbb{R}) : a_{i,j} = 0 \iff \{i, j\} \in E(\overline{G})\}.$$

Let $A \in \mathcal{S}(G)$. Then $A + B \in \mathcal{S}(G)$ when $\|B\|$ is small enough.

The tangent space of $F(B, K)$ at (O, O) with respect to B is $\mathcal{S}^{\mathrm{cl}}(G)$.

Isospectral perturbation

$$F(B, K) = e^{-K}Ae^{K} + B$$

The function e^K is a bijection between

$$\{\text{skew-symmetric matrices nearby } O\} \to \{\text{orthogonal matrices nearby } I\}$$

for real matrices.

The tangent space of $F(B, Q)$ at (O, O) with respect to Q is

$$\{-KA + AK : K \in \mathrm{Skew}_n(\mathbb{R})\}.$$

Strong spectral property, transversality, and surjectivity

Definition

Let A be a real symmetric matrix. Then A has the strong spectral property (SSP) if $X = O$ is the only real symmetric matrix that satisfies

$$A \circ X = I \circ X = [A, X] = O.$$

Let $F(B, K) = e^{-K} A e^{K} + B$. Then the following are equivalent:

1. A has the SSP.
2. $\mathcal{S}^{\mathrm{cl}}(G) + \{-KA + AK : K \in \mathrm{Skew}_n(\mathbb{R})\} = \mathrm{Sym}_n(\mathbb{R})$.
3. The derivative $\dot{F}$ is surjective.

Illustration of the supergraph lemma

$$F(B,K) = e^{-K}Ae^{K} + B$$

$$M = \begin{bmatrix} 1 & \epsilon & 0 & 0 \\ \epsilon & 2 & \epsilon & 0 \\ 0 & \epsilon & 3 & \epsilon \\ 0 & 0 & \epsilon & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} \sim 1 & \epsilon & 0 & 0 \\ \epsilon & \sim 2 & \epsilon & 0 \\ 0 & \epsilon & \sim 3 & \epsilon \\ 0 & 0 & \epsilon & \sim 4 \end{bmatrix}$$

same pattern

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

same spectrum

For whatever pattern nearby, there is A' with $\operatorname{spec}(A') = \operatorname{spec}(A)$.

They must be true, right?

Let $A \in \mathcal{S}(G)$ with the SSP. People *believed* that ...

- For any set of real numbers Λ' nearby $\operatorname{spec}(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $\operatorname{spec}(A') = \Lambda'$.
- For any refinement $\mathbf{m}'$ of $\mathbf{m}(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $\mathbf{m}(A') = \mathbf{m}'$.
- For any $k > q(A)$, there is a matrix $A' \in \mathcal{S}(G)$ with $q(A') = k$.

Let $A \in \mathcal{Q}(P)$ be a nilpotent matrix with the nSSP. People *knew* that ...

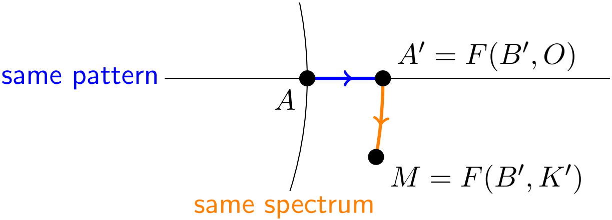
- For any set of complex numbers Λ' (invariant under conjugation) nearby $\{0, \ldots, 0\}$, there is a matrix $A' \in \mathcal{Q}(P)$ with $\operatorname{spec}(A') = \Lambda'$.

nSSP = the condition of the nilpotent-centralizer method

Bifurcation lemma

Theorem (FHLS 2022)

Let $A \in \mathcal{S}(G)$ with the SSP. Then for any set of real numbers Λ' nearby $\operatorname{spec}(A)$, there is a matrix A' with $\operatorname{spec}(A') = \Lambda'$.



$$F(B, K) = e^{-K}(A + B)e^{K}$$

The nSSP

Definition

Let A be a real matrix. Then A has the non-symmetric strong spectral property (nSSP) if $X = O$ is the only real matrix that satisfies

$$A \circ X = [A, X^\top] = O.$$

Let $\mathcal{Q}^{\mathrm{v}}(P)$ be the set of matrices with the same zero entries as P.

Let $F(B, Q) = Q^{-1}(A + B)Q$, where $B \in \mathcal{Q}^{\mathrm{v}}(P)$. Then the following are equivalent:

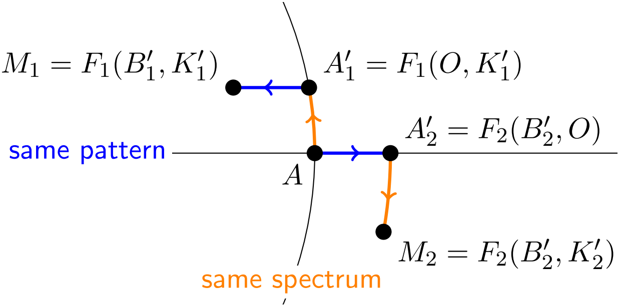
1. A has the nSSP.
2. $\mathcal{Q}^{\mathrm{v}}(P) + \{-LA + AL : L \in \mathrm{Mat}_n(\mathbb{R})\} = \mathrm{Mat}_n(\mathbb{R})$.
3. The derivative $\dot{F}$ is surjective.

Bifurcation lemma for non-symmetric matrices

Theorem (FHLS 2022)

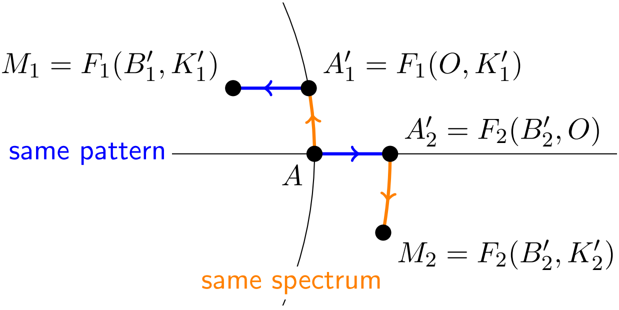
Let $A \in \mathcal{Q}(P)$ *with the nSSP for some sign pattern* P*. Then for any set of complex numbers* Λ' *(invariant under conjugation) nearby* $\operatorname{spec}(A)$*, there is a matrix* $A' \in \mathcal{Q}(P)$ *with* $\operatorname{spec}(A') = \Lambda'$*.*

Supergraph lemma: $F_1(B,K) = e^{-K}Ae^{K} + B$



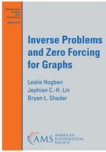
Bifurcation lemma: $F_2(B,K) = e^{-K}(A+B)e^{K}$

Supergraph lemma: $F_1(B,K) = e^{-K}Ae^{K} + B$



Bifurcation lemma: $F_2(B,K) = e^{-K}(A+B)e^{K}$

For more IEP-G...



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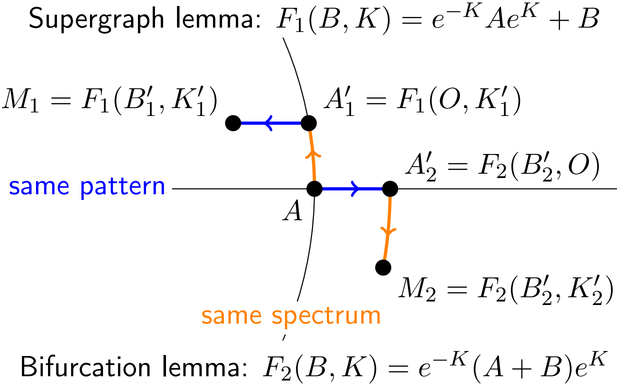

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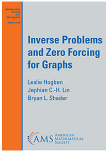

Thanks!

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