

Using new zero forcing parameters to guarantee the Strong Arnold Property

Jephian C.-H. Lin

Department of Mathematics, Iowa State University

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Maximum nullity

- ▶ For a simple graph G , let $\mathcal{S}(G)$ be the family of **real, symmetric** matrices $A = [a_{i,j}]$ such that

$$a_{i,j} \begin{cases} \neq 0 & \text{if } i \sim j, i \neq j \\ = 0 & \text{if } i \not\sim j, i \neq j \\ \in \mathbb{R} & \text{if } i = j \end{cases}$$

- ▶ The **maximum nullity** is the largest possible nullity happens in $\mathcal{S}(G)$. That is,

$$M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\}$$

Example of $M(G)$

- ▶ Let $G = P_5$. Then the matrices in $\mathcal{S}(G)$ looks like

$$\begin{bmatrix} ? & * & 0 & 0 & 0 \\ * & ? & * & 0 & 0 \\ 0 & * & ? & * & 0 \\ 0 & 0 & * & ? & * \\ 0 & 0 & 0 & * & ? \end{bmatrix}$$

- ▶ The Laplacian matrix is in $\mathcal{S}(G)$ and has nullity 1.
- ▶ $M(G) = 1$.

Example of $M(G)$

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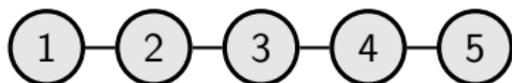
$$\begin{bmatrix} ? & * & 0 & 0 & 0 \\ * & ? & * & 0 & 0 \\ 0 & * & ? & * & 0 \\ 0 & 0 & * & ? & * \\ \del{0} & \del{0} & \del{0} & \del{*} & \del{?} \end{bmatrix}$$

- ▶ The Laplacian matrix is in $\mathcal{S}(G)$ and has nullity 1.
- ▶ $M(G) = 1$.

Zero forcing number

- ▶ Zero forcing game: All vertices are either **blue** or **white**. At the beginning, pick some vertices to be blue, then trying to use the **color change rule** repeatedly to force all vertices to turn blue.
- ▶ Color change rule: if x is a **blue** vertex and y is the only **white** neighbor of x , then y turns **blue**. (Denoted by $x \rightarrow y$.)
- ▶ If beginning with a set S of blue vertices and all vertices can turn blue, then S is called a **zero forcing set**.
- ▶ The **zero forcing number** $Z(G)$ is the minimum cardinality of zero forcing sets on G .

Example of $Z(G)$



- ▶ Initially set 1 as blue.
- ▶ $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 4$, $4 \rightarrow 5$.
- ▶ One blue vertex is enough, and also required, so $Z(G) = 1$.

Example of $Z(G)$



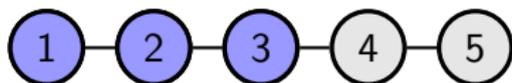
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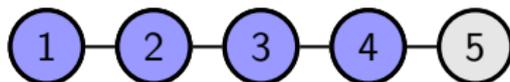
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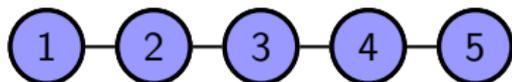
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$M(G) \leq Z(G)$

- ▶ For all graph G , $M(G) \leq Z(G)$ [AIM (2008)].
- ▶ $M(G) = Z(G)$ for small graphs up to 7 vertices. [DGHMST (2010)]
- ▶ $M(G) = Z(G)$ for trees, cycles, hypercube, block-clique graphs ...

Strong Arnold Property

- ▶ A real symmetric matrix A is said to have the **Strong Arnold Property** (SAP) if the only real symmetric matrix X that satisfies

$$\begin{cases} A \circ X = O \\ I \circ X = O \\ AX = O \end{cases}$$

is $X = O$. Here \circ is the Hadamard (entrywise) product.

- ▶ If A is nonsingular, then A has the SAP.
- ▶ If $A \in \mathcal{S}(K_n)$, then A has the SAP.

Example of not having the SAP

Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $A \circ X = I \circ X = O$ and $AX = O$, so A does not have the SAP.

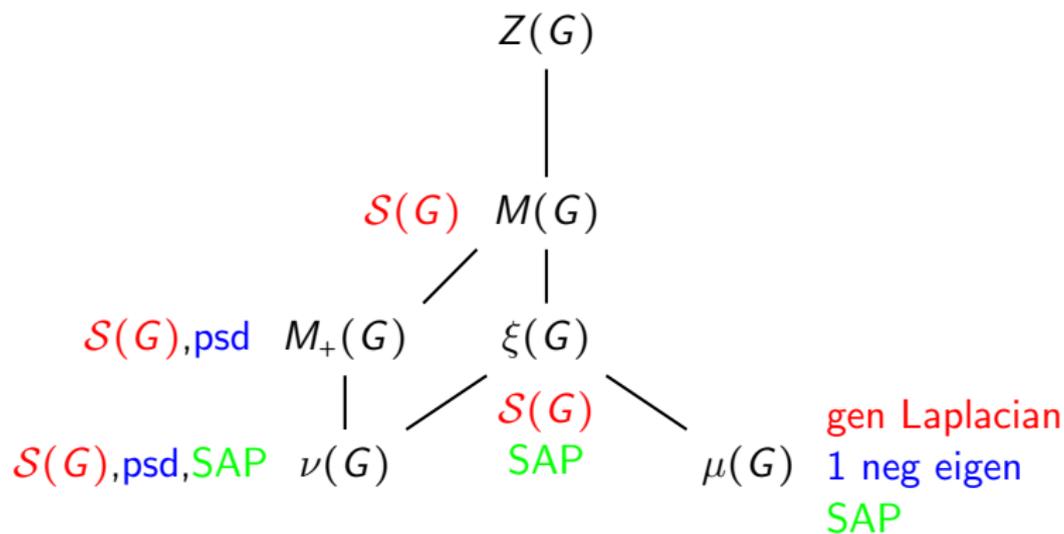
Colin de Verdière parameter $\mu(G)$

- ▶ For a simple graph G , the **Colin de Verdière parameter** $\mu(G)$ [Colin de Verdière (1990)] is the maximum nullity over matrices A such that
 - ▶ $A \in \mathcal{S}(G)$ and all off-diagonal entries are zero or negative. (Called **generalized Laplacian**.)
 - ▶ A has **exactly one negative eigenvalue** (counting multiplicity).
 - ▶ A has **the SAP**.
- ▶ Characterizations:
 - ▶ $\mu(G) \leq 1$ iff G is a disjoint union of paths. (No K_3 minor)
 - ▶ $\mu(G) \leq 2$ iff G is outer planar. (No $K_4, K_{2,3}$ minor)
 - ▶ $\mu(G) \leq 3$ iff G is planar. (No $K_5, K_{3,3}$ minor)
- ▶ It is conjectured that $\mu(G) + 1 \geq \chi(G)$.

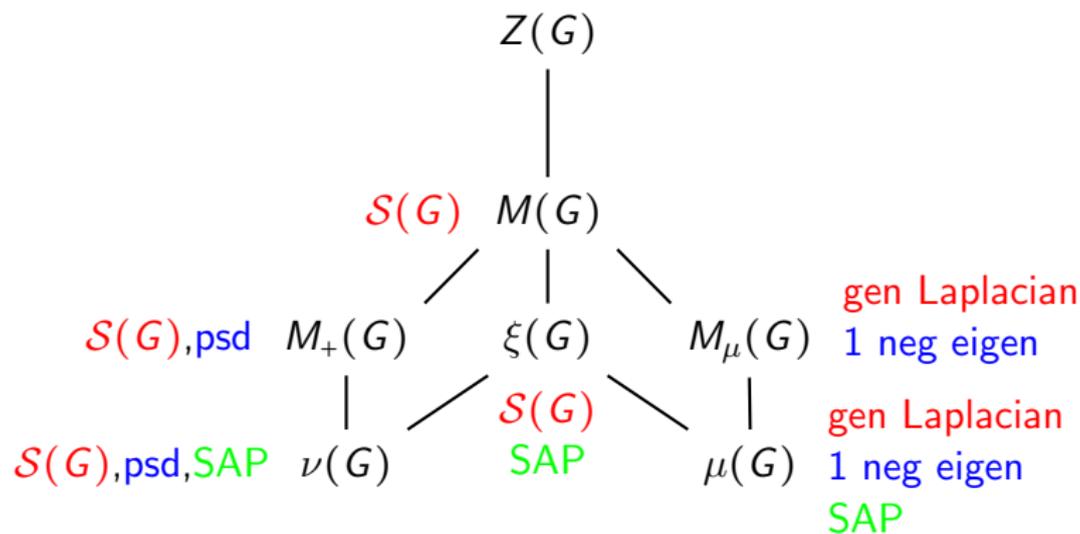
Other Colin de Verdière type parameters

- ▶ $\xi(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ has the SAP}\}$
- ▶ $\nu(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ is PSD}, A \text{ has the SAP}\}$
- ▶ For Colin de Verdière type parameters $\beta \in \{\mu, \nu, \xi\}$, they are all **minor monotone**. That is, $\beta(H) \leq \beta(G)$ if H is a minor of G . [C (1990), C (1998), BFH (2005)]
- ▶ By graph minor theorem, $\beta(G) \leq k$ if and only if G does not contain a family of **finite** graphs as minors. (Called forbidden minors.)

Colin de Verdière type parameters



Colin de Verdière type parameters



M and ξ

- ▶ M is not minor monotone, but ξ is; and

$$\xi(G) \leq M(G).$$

- ▶ For a parameter β , we can consider

$$\lfloor \beta \rfloor(G) := \min\{\beta(H) : G \text{ is a minor of } H\}.$$

- ▶ For all graph $M(G) \leq Z(G)$, so $\xi(G) \leq \lfloor M \rfloor(G) \leq \lfloor Z \rfloor(G)$.
- ▶ $\lfloor Z \rfloor(G)$ can be computed. [BBFHHSvdDvdH (2013)]
- ▶ How to compute $\xi(G)$?
- ▶ For what graph $\xi(G) = M(G)$ or $\xi(G) = \lfloor Z \rfloor(G)$?

Graph structure guarantees the SAP?

- ▶ If $G = K_n$, then every matrix $A \in \mathcal{S}(G)$ has the SAP.
- ▶ If G is connected such that \overline{G} is a **matching**, then every matrix $A \in \mathcal{S}(G)$ has the SAP. [BFH (2005)]
- ▶ If G is connected such that \overline{G} is a **forest**, then every matrix $A \in \mathcal{S}(G)$ has the SAP.
- ▶ The **SAP zero forcing number** Z_{SAP} will be defined later.

Theorem (JL 2016+)

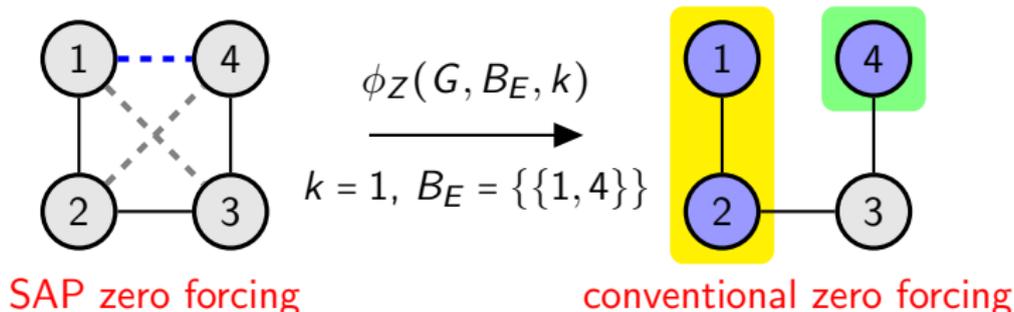
*If $Z_{\text{SAP}}(G) = 0$, then every matrix $A \in \mathcal{S}(G)$ has the SAP.
Therefore, $\xi(G) = M(G)$, $M_+(G) = \nu(G)$, and $M_\mu(G) = \mu(G)$.*

SAP zero forcing

- ▶ In an SAP zero forcing game, **every non-edge** has color either **blue** or white.
- ▶ If B_E is the set of blue non-edges, the **local game** on a given vertex k is a **conventional** zero forcing game on G , with blue vertices

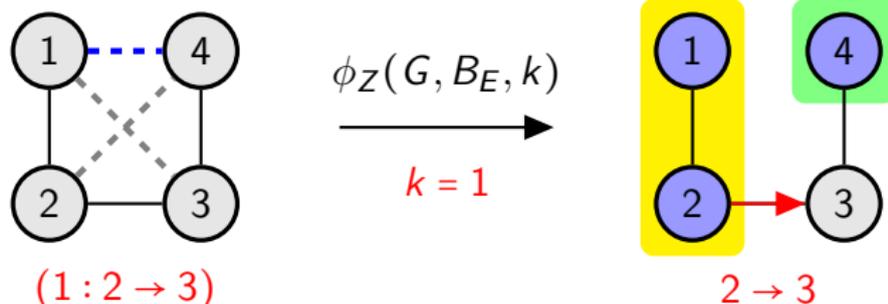
$$\phi_k(G, B_E) := N_G[k] \cup N_{(B_E)}(k).$$

The local game is denoted by $\phi_Z(G, B_E, k)$.



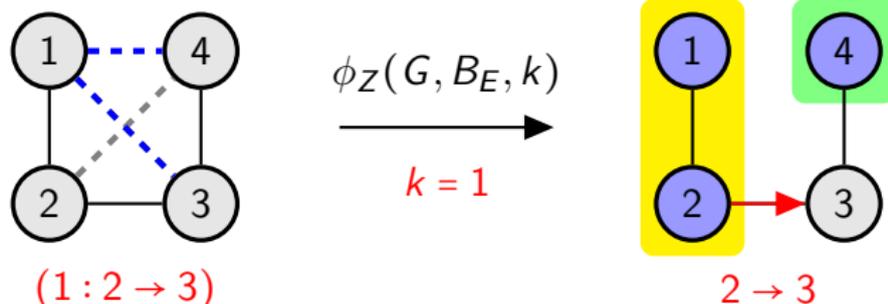
SAP zero forcing

- ▶ Color change rule- Z_{SAP} :
 - ▶ Forcing triple $(k : i \rightarrow j)$: If $i \rightarrow j$ in $\phi_Z(G, B_E, k)$, then $\{j, k\}$ turns blue.
 - ▶ Odd cycle rule $(i \rightarrow C)$: Let G_W be the graph whose edges are the white non-edges. If $G_W[N_G(i)]$ contains a component that is a odd cycle C . Then $E(C)$ turns blue.
- ▶ $Z_{SAP}(G)$ is the minimum number of blue non-edges such that all non-edges can turn blue eventually by CCR- Z_{SAP} .



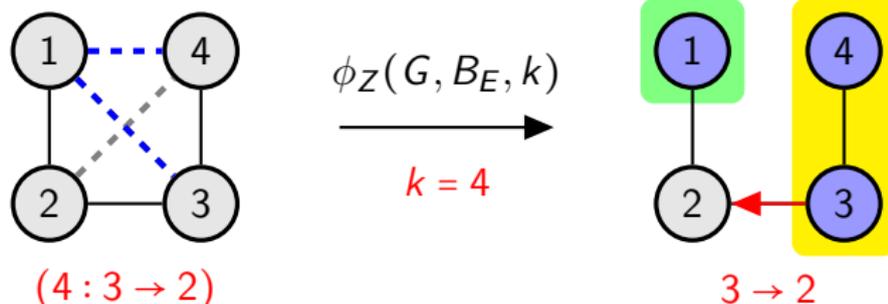
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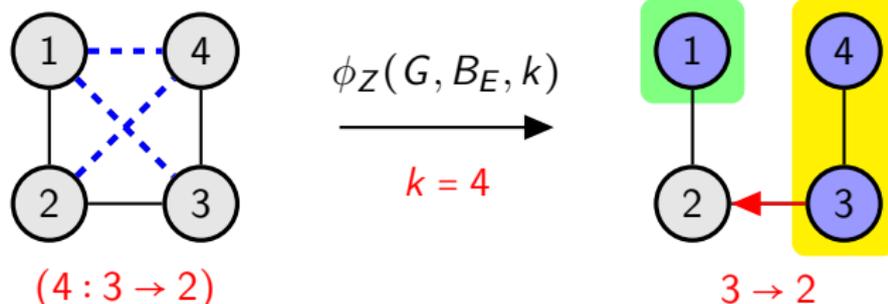
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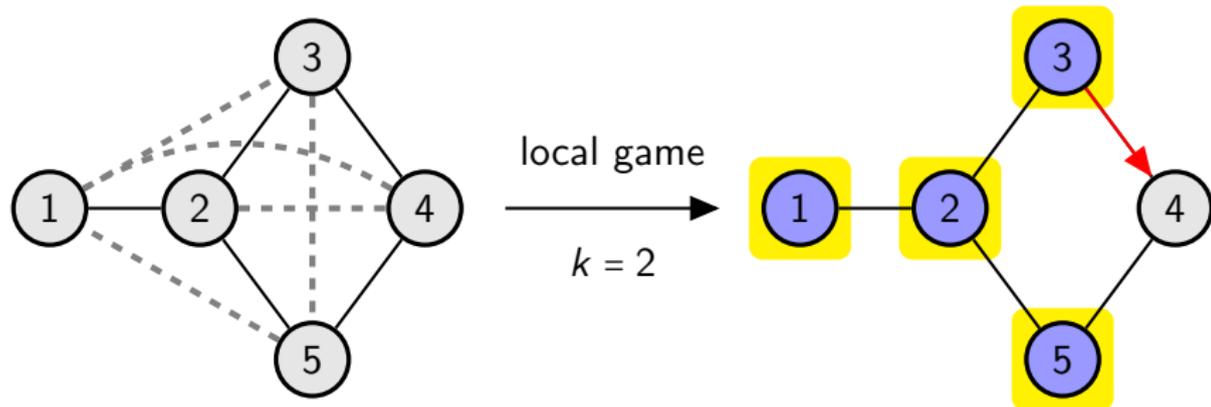


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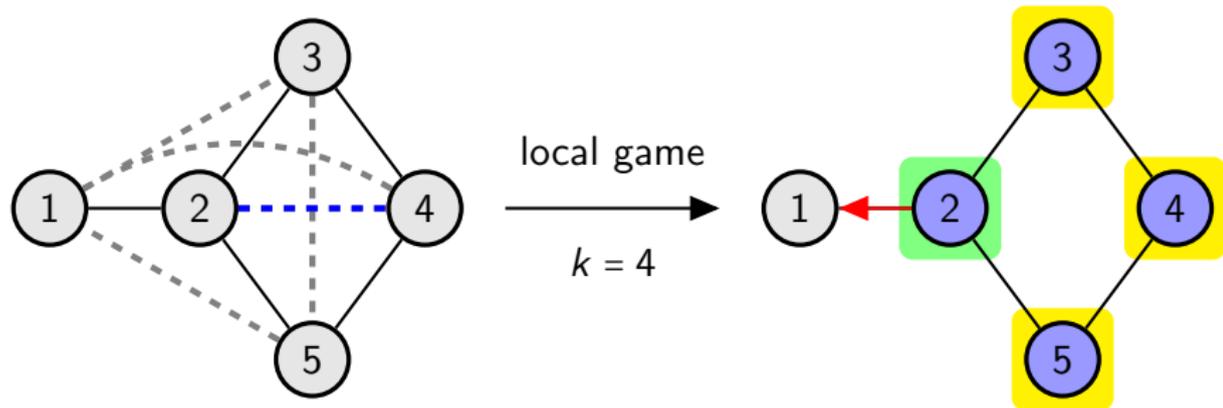


Example of $Z_{\text{SAP}}(G) = 0$



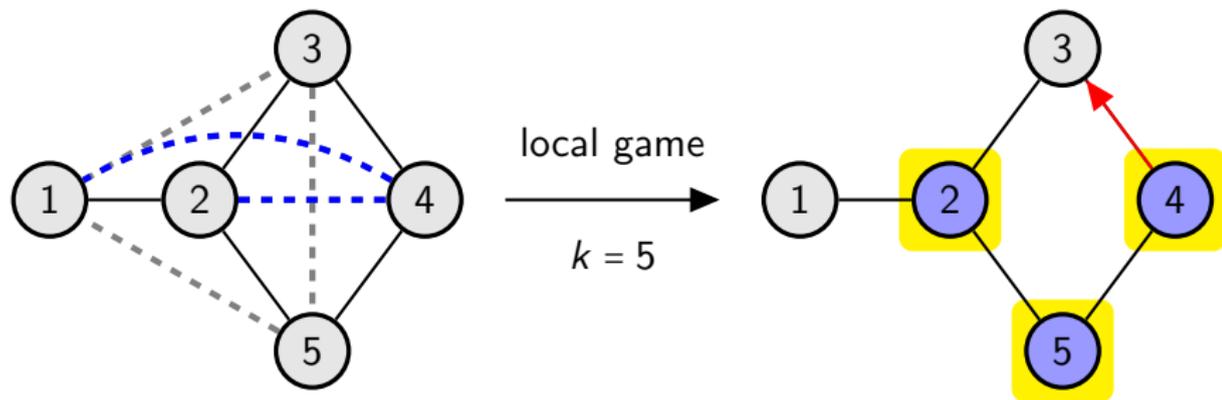
Step	Forcing triple	Forced non-edge
1	$(2 : 3 \rightarrow 4)$	$\{2, 4\}$
2	$(4 : 2 \rightarrow 1)$	$\{4, 1\}$
3	$(5 : 4 \rightarrow 3)$	$\{5, 3\}$
4	$(3 : 2 \rightarrow 1)$	$\{3, 1\}$
5	$(5 : 2 \rightarrow 1)$	$\{5, 1\}$

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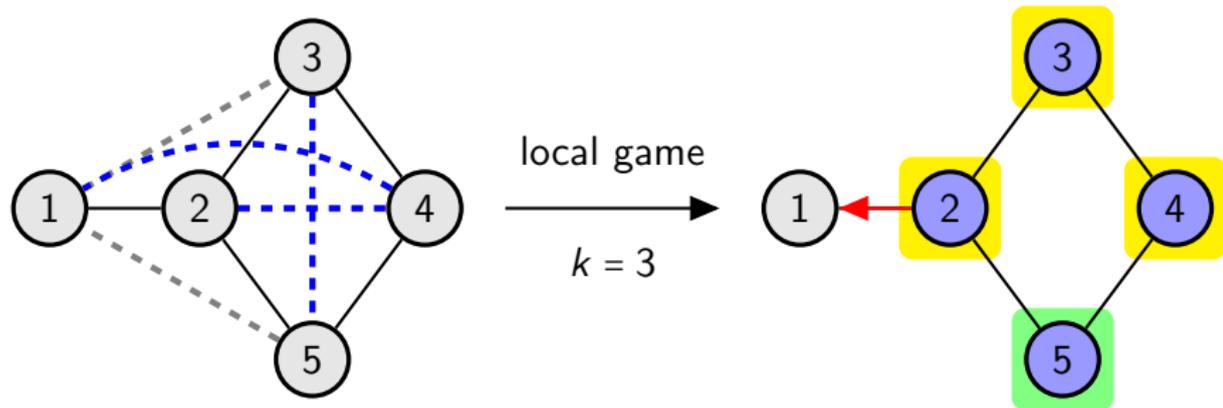
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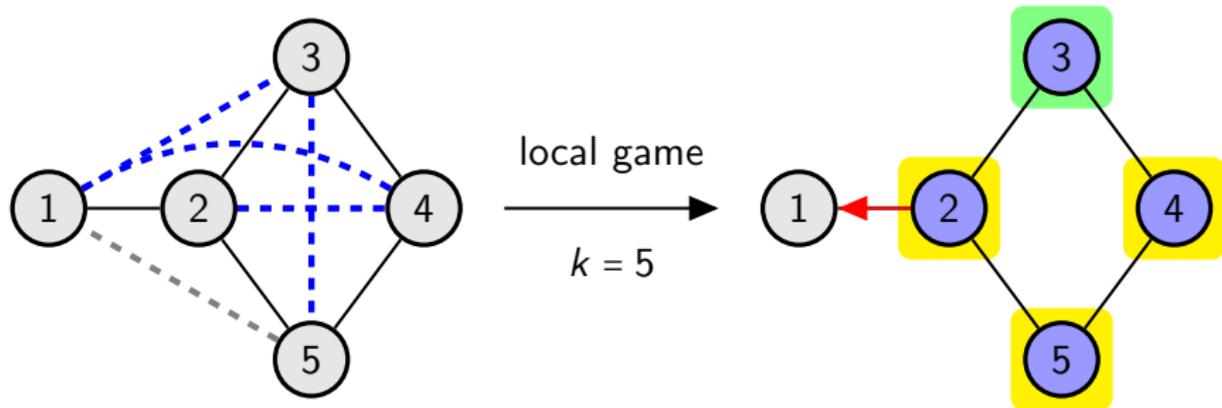
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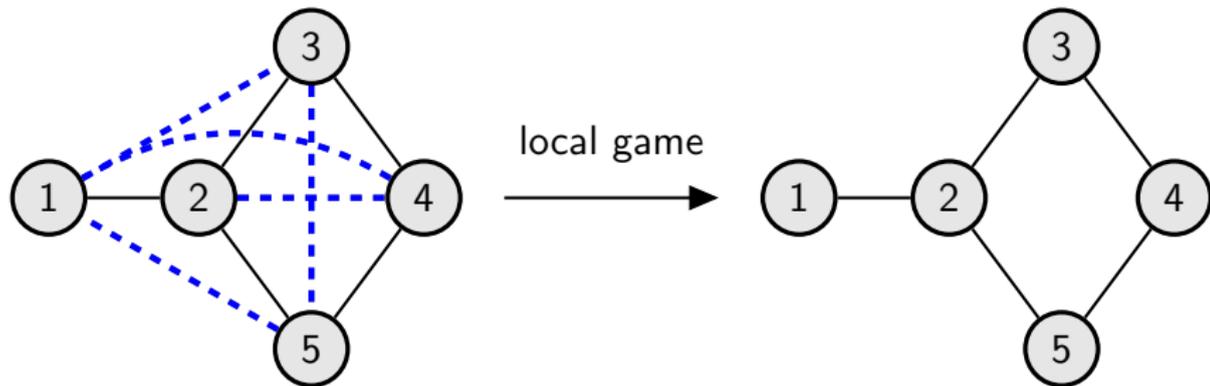
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Theorem (JL 2016+)

If $Z_{\text{SAP}}(G) = 0$, then every matrix $A \in \mathcal{S}(G)$ has the SAP.

Therefore, $\xi(G) = M(G)$, $M_+(G) = \nu(G)$, and $M_\mu(G) = \mu(G)$.

How to test the SAP?

- ▶ Let G be a graph and $A \in \mathcal{S}(G)$ with \mathbf{v}_j its j -th column vector. Let $\overline{m} = |E(\overline{G})|$.
- ▶ The **SAP matrix** Ψ of A is an $n^2 \times \overline{m}$ matrix with
 - ▶ row indexed by (i, j) with $i, j \in \{1, \dots, n\}$
 - ▶ column indexed by $\{i, j\} \in E(\overline{G})$
 - ▶ the $\{i, j\}$ -th column of Ψ is

$$(\mathbf{0}, \dots, \mathbf{0}, \underbrace{\mathbf{v}_j}_{i\text{-th block}}, \mathbf{0}, \dots, \mathbf{0}, \underbrace{\mathbf{v}_i}_{j\text{-th block}}, \mathbf{0}, \dots, \mathbf{0})^\top$$

- ▶ A has the SAP if and only if Ψ is **full-rank**.

Example of the SAP matrix: forcing triples

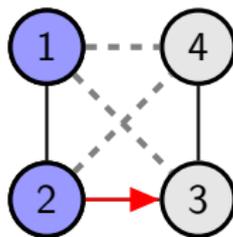
- ▶ Recall the SAP: $A \circ X = I \circ X = AX = O \implies X = O$.
- ▶ Let $G = P_4$ and $A \in \mathcal{S}(G)$ with \mathbf{v}_j its j -th column vector.

$$AX = \begin{bmatrix} d_1 & a_1 & 0 & 0 \\ a_1 & d_2 & a_2 & 0 \\ 0 & a_2 & d_3 & a_3 \\ 0 & 0 & a_3 & d_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & x_{\{1,3\}} & x_{\{1,4\}} \\ 0 & 0 & 0 & x_{\{2,4\}} \\ x_{\{1,3\}} & 0 & 0 & 0 \\ x_{\{1,4\}} & x_{\{2,4\}} & 0 & 0 \end{bmatrix} = O.$$

- ▶ This is equivalent to

$$\begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} x_{\{1,3\}} \\ x_{\{1,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \Psi \begin{bmatrix} x_{\{1,3\}} \\ x_{\{1,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \mathbf{0}.$$

$$\begin{array}{ccc|ccc}
 0 & 0 & 0 & & & \\
 1 & 0 & 0 & & & \\
 -1 & 1 & 0 & & & \\
 1 & -1 & 0 & & & \\
 \hline
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 1 & & & \\
 0 & 0 & -1 & & & \\
 \hline
 -1 & 0 & 0 & & & \\
 1 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 \hline
 0 & -1 & 1 & & & \\
 0 & 1 & -1 & & & \\
 0 & 0 & 1 & & & \\
 0 & 0 & 0 & & &
 \end{array}
 \begin{array}{l}
 X_{\{1,3\}} \\
 X_{\{1,4\}} \\
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 \end{array}$$

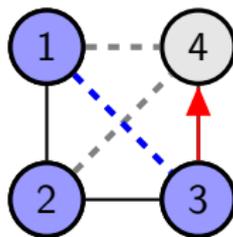


(1:2 → 3)

At block 1,
look at row 2,

then the only nonzero entry is at column {1,3}.

$$\begin{array}{ccc|ccc}
 0 & 0 & 0 & & & \\
 1 & 0 & 0 & & & \\
 -1 & 1 & 0 & & & \\
 1 & -1 & 0 & & & \\
 \hline
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 1 & & & \\
 0 & 0 & -1 & & & \\
 \hline
 -1 & 0 & 0 & & & \\
 1 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 \hline
 0 & -1 & 1 & & & \\
 0 & 1 & -1 & & & \\
 0 & 0 & 1 & & & \\
 0 & 0 & 0 & & &
 \end{array}
 \begin{array}{l}
 X_{\{1,3\}} \\
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 \end{array}$$

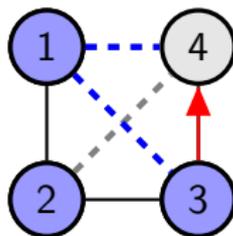


(1 : 3 → 4)

At block 1,
look at row 3,

then the only nonzero entry is at column {1,4}.

$$\begin{array}{ccc|ccc}
 0 & 0 & 0 & & & \\
 1 & 0 & 0 & & & \\
 -1 & 1 & 0 & & & \\
 1 & -1 & 0 & & & \\
 \hline
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 1 & & & \\
 0 & 0 & -1 & & & \\
 \hline
 -1 & 0 & 0 & & & \\
 1 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 \hline
 0 & -1 & 1 & & & \\
 0 & 1 & -1 & & & \\
 0 & 0 & 1 & & & \\
 0 & 0 & 0 & & &
 \end{array}
 \begin{array}{l}
 X_{\{1,3\}} \\
 X_{\{1,4\}} \\
 X_{\{2,4\}}
 \end{array}$$

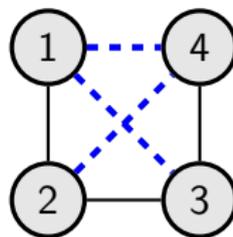


(2:3 → 4)

At block 2,
look at row 3,

then the only nonzero entry is at column {2,4}.

$$\begin{array}{ccc|ccc}
 0 & 0 & 0 & & & \\
 1 & 0 & 0 & & & \\
 -1 & 1 & 0 & & & \\
 1 & -1 & 0 & & & \\
 \hline
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 1 & & & \\
 0 & 0 & -1 & & & \\
 \hline
 -1 & 0 & 0 & & & \\
 1 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 0 & 0 & 0 & & & \\
 \hline
 0 & -1 & 1 & & & \\
 0 & 1 & -1 & & & \\
 0 & 0 & 1 & & & \\
 0 & 0 & 0 & & &
 \end{array}
 \left[\begin{array}{l} X_{\{1,3\}} \\ X_{\{1,4\}} \\ X_{\{2,4\}} \end{array} \right]$$



Example of the SAP matrix: odd cycle rules

- ▶ Recall the SAP: $A \circ X = I \circ X = AX = O \implies X = O$.
- ▶ Let $G = K_{1,3}$ and $A \in \mathcal{S}(G)$ with \mathbf{v}_j its j -th column vector.

$$AX = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_{\{2,3\}} & x_{\{2,4\}} \\ 0 & x_{\{2,3\}} & 0 & x_{\{3,4\}} \\ 0 & x_{\{2,4\}} & x_{\{3,4\}} & 0 \end{bmatrix} = O.$$

- ▶ This is equivalent to

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{v}_3 & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_2 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_3 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} x_{\{2,3\}} \\ x_{\{3,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \Psi \begin{bmatrix} x_{\{2,3\}} \\ x_{\{3,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \mathbf{0}.$$

$$\Psi = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{v}_3 & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_2 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_3 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \\ 2 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{full rank.}$$

Proof of the main theorem

Proof.

- ▶ Assume $\Psi \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (x_e)_{e \in E(\overline{G})}$.
- ▶ e is white means x_e is possibly non-zero; e is blue means x_e is zero.
- ▶ Starting with all white:
 - ▶ $(k : i \rightarrow j)$ implies $x_{\{j,k\}} = 0$.
 - ▶ $(i \rightarrow C)$ implies $x_e = 0$ for all $e \in E(C)$.
- ▶ When all non-edges are blue, it means $\mathbf{x} = \mathbf{0}$ is the only right kernel. So Ψ is full-rank.



Computational results

How many graphs has the property $Z_{\text{SAP}}(G) = 0$? The table shows for fixed n the proportion of graphs with $Z_{\text{SAP}}(G)$ in all connected graphs. (Isomorphic graphs count only once.)

n	$Z_{\text{SAP}} = 0$
1	1.0
2	1.0
3	1.0
4	1.0
5	0.86
6	0.79
7	0.74
8	0.73
9	0.76
10	0.79

Applications

Theorem (JL 2016+)

For all graph G up to 7 vertices, $\xi(G) = \lfloor Z \rfloor(G)$.

Proof.

By Sage program, one of the following will happen:

- ▶ $Z_{\text{SAP}}(G) = 0 \implies \xi(G) = \lfloor Z \rfloor(G)$.
- ▶ G is a tree $\implies \xi(G) = \lfloor Z \rfloor(G)$.
- ▶ $\lfloor Z \rfloor(G) = M(G) - Z_{\text{vc}}(G) \implies \xi(G) = \lfloor Z \rfloor(G)$.
- ▶ $\lfloor Z \rfloor(G) = \eta(G) - 1 \implies \xi(G) = \lfloor Z \rfloor(G)$.
- ▶ $\lfloor Z \rfloor(G) = 3$ and G contains a T_3 -minor $\implies \xi(G) = \lfloor Z \rfloor(G)$.



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Thank you!

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